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A Convergence-Guaranteed Nonlinear Optimization Framework for the Modified Wilson EOQ Model under Time-Dependent Demand

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Abstract. This study develops an extended Economic Order Quantity (EOQ) model for inventory systems under linearly increasing demand. Unlike the classical Wilson model based on constant demand assumptions, the proposed approach incorporates dynamic demand behavior within a nonlinear programming framework. The solution methodology applies the Pshenichny linearization method, transforming the original problem into auxiliary quadratic programming subproblems with guaranteed convergence properties.

A numerical application demonstrates the effectiveness of the proposed model in determining optimal order quantities under changing demand conditions. The obtained results show stable convergence of the algorithm and provide optimal replenishment policies that minimize total inventory costs under time-varying demand. The numerical experiments confirm the computational efficiency of the proposed approach and illustrate its applicability to inventory systems characterized by dynamic demand patterns.

The study contributes to the inventory optimization literature by introducing a convergence-guaranteed nonlinear optimization approach for dynamic-demand inventory management problems.

Keywords: Wilson EOQ model, inventory optimization, linear demand, nonlinear programming, quadratic programming, dual optimization, retail supply chain.

1. Introduction

Inventory optimization is a critical component of supply chain and retail management, where organizations seek to balance ordering, holding, and shortage costs while maintaining satisfactory customer service levels. Effective inventory control directly influences operational efficiency, customer satisfaction, and overall profitability. As supply chains become increasingly complex and market conditions more volatile, the development of accurate inventory planning models has become an important area of operations research.

Among the various inventory control approaches, the Economic Order Quantity (EOQ) model remains one of the most widely used analytical frameworks in inventory management and supply chain planning. Originally formalized by Wilson (1934), the EOQ model provides a simple yet powerful method for determining optimal replenishment quantities under given operational conditions. Due to its simplicity, analytical tractability, and managerial relevance, it has become a foundational tool and has inspired a wide range of theoretical and practical extensions. However, one of the most restrictive assumptions of the classical EOQ model is the assumption of constant demand. In real-world retail and supply chain environments, demand frequently varies over time due to seasonality, market trends, promotional activities, and changes in consumer behavior, which may significantly affect inventory performance and decision quality.

To address these limitations, researchers have developed numerous EOQ extensions incorporating dynamic demand patterns, deterioration effects, shortages, and operational constraints (Liu et al., 2014; Manna et al., 2017; Sarkar, 2013; Taleizadeh, 2017). Among these developments, inventory models with time-dependent demand have attracted considerable attention because they provide a more realistic representation of inventory dynamics and replenishment decisions.

Within this research stream, Zeng et al. (2019) proposed a modified Wilson EOQ model with a linearly increasing demand function:

$$\theta(t) = \mu + \omega\,(t - 1),\ \ t \in (1, T)$$

where μ denotes the average daily demand and ω represents the demand growth rate over the planning horizon. The resulting inventory optimization problem was formulated as a nonlinear cost minimization model subject to ordering constraints.

Although the model proposed by Zeng et al. (2019) represents an important advancement, its solution methodology relies on simplifying assumptions such as equal inter-order intervals and equal order quantities. Furthermore, nonlinear terms are approximated through truncated series expansions, while rigorous convergence analysis and approximation error bounds are not provided. Similar limitations can also be observed in several related studies employing approximation-based analytical techniques (Battini et al., 2014; Liberopoulos et al., 2010; Broyles et al., 2010; Sana, 2011).

To address these limitations, this study develops a convergence-guaranteed numerical optimization framework for the modified Wilson EOQ model under linearly varying demand. The proposed methodology combines Pshenichny linearization procedures with auxiliary quadratic programming techniques to solve the original nonlinear problem directly (Pshenichny, 1983; Pshenichny & Danilin, 1975). Unlike previous approximation-based approaches, the proposed framework avoids restrictive simplifying assumptions and provides stable, computationally efficient, and theoretically justified solutions. The contribution of this study lies primarily in the development of a robust numerical solution methodology applicable not only to the model of Zeng et al. (2019) but also to a broader class of nonlinear EOQ-type inventory optimization problems with time-dependent demand and structural constraints.

2. Literature Review

2.1. Classical EOQ and Foundational Inventory Models

Inventory optimization has long been a central topic in operations research due to its direct impact on cost efficiency, service level performance, and supply chain stability. Early studies laid the foundation for understanding demand behavior and inventory dynamics under simplified assumptions. Among these, the classical Economic Order Quantity (EOQ) model, formalized by Wilson (1934), remains one of the most influential analytical tools for determining optimal order quantities under deterministic conditions.

The EOQ framework is widely valued for its simplicity and tractability; however, it is built on restrictive assumptions, particularly constant demand and stable cost parameters. While such assumptions allow closed-form solutions, they limit applicability in real-world environments where demand fluctuates and uncertainty is inherent.

Early attempts to relax these assumptions include Schwartz (1970), who examined inventory behavior under perturbed demand conditions. Although this work introduced dynamic considerations, it primarily focused on demand variation effects rather than embedding them into a fully optimized EOQ structure. Similarly, Dubelaar et al. (2001) empirically demonstrated the relationship between inventory levels and retail performance, but their approach remained largely observational rather than optimization-based.

Overall, these foundational studies highlight the importance of demand variability but do not yet provide fully integrated dynamic EOQ formulations.

2.2. Dynamic-Demand and Retail Inventory Systems

Building on early foundational work, subsequent research shifted toward incorporating explicit demand variability into inventory models. Boute et al. (2007) developed integrated production-inventory systems aimed at reducing demand amplification effects, highlighting the interaction between upstream production decisions and downstream demand stability. However, their model primarily addresses system coordination rather than EOQ optimization under time-varying demand.

Liu et al. (2014) and Manna et al. (2017) further extended inventory systems by incorporating multiple demand classes and advertisement-dependent demand structures. These studies improve realism by accounting for behavioral and policy-driven demand fluctuations. Nevertheless, their formulations often rely on specific structural assumptions that limit generalization across broader EOQ-type systems.

Empirical studies such as Ortega et al. (2015) and Bendoly et al. (2018) emphasize operational consistency and consumer behavior in retail environments, reinforcing the importance of demand variability. However, these works are primarily descriptive and provide limited contribution to optimization methodology.

Recent contributions (e.g., Luo et al., 2024; Kankoon & Amornsawadwatana, 2024; Jama & Mohamud, 2024) further confirm that adaptability and resilience significantly enhance supply chain performance. Despite their relevance, these studies focus more on system outcomes rather than providing analytical optimization frameworks.

2.3. Perishable, Imperfect, and Deteriorating Inventory Systems

A significant extension of EOQ literature involves inventory systems with deterioration, perishability, and production imperfections. Sarkar (2013) introduced a two-echelon production-inventory model incorporating probabilistic deterioration, demonstrating that neglecting product decay can lead to suboptimal replenishment strategies.

Similarly, Nobil and Taleizadeh (2016) examined imperfect production systems with rework mechanisms, highlighting how manufacturing defects influence optimal inventory decisions. While

these models add realism, they typically assume predefined demand structures and do not fully integrate time-dependent demand dynamics.

Environmental and sustainability considerations were introduced by Wahab et al. (2011), who incorporated green supply chain factors into EOQ coordination models. Fleisch and Tellkamp (2005) further showed that inventory inaccuracies significantly affect system performance, particularly in data-driven environments. However, these studies primarily extend EOQ structures in specific directions rather than providing unified frameworks for multiple uncertainties.

2.4. Stochastic, Markovian, and Uncertainty-Based Inventory Models

Another major research direction focuses on stochastic demand modeling and uncertainty representation. Isotupa (2006) and Yin et al. (2002) applied Markovian decision processes to inventory systems, enabling modeling of demand transitions and probabilistic state changes. These approaches significantly improve realism but often lead to high computational complexity

Broyles et al. (2010) introduced statistical approximation techniques for transient inventory systems, particularly in healthcare applications. While computationally efficient, such approximations may compromise solution accuracy and stability.

More recent works, such as Budd and Taylor (2019), derive analytical bounds for compound renewal demand processes, offering valuable theoretical insights. However, these bounds are often difficult to translate into implementable operational policies.

Robust and multistage optimization approaches (Minoux, 2018) address uncertainty more systematically, but they typically require heavy computational resources. Similarly, studies on disruption management (Paul et al., 2018; Konstantaras et al., 2019) emphasize resilience but do not always integrate with classical EOQ formulations in a unified manner.

2.5. Nonlinear Optimization and Mathematical Programming Methods

Many advanced inventory models naturally lead to nonlinear optimization problems that cannot be solved analytically. Liberopoulos et al. (2010) examined EOQ sensitivity under nonlinear cost structures, showing that small parameter changes can significantly affect optimal policies. However, their approach remains model-specific and does not provide a general solution framework.

Battini et al. (2014) extended EOQ models by incorporating sustainability constraints, introducing multi-objective considerations into inventory optimization. Although this enhances practical relevance, the resulting models increase computational complexity.

From a methodological perspective, Pshenichny and Danilin (1975) and Pshenichny (1983) developed foundational linearization and extremal problem-solving techniques that allow transformation of nonlinear optimization problems into more tractable forms. These methods provide a strong theoretical basis for numerical convergence and stability analysis.

Additional contributions in logistics optimization (Nestorenko et al., 2017; Slesarenko and Nestorenko, 2014; Ortolani et al., 2011) demonstrate the applicability of mathematical programming across supply chain systems. However, many of these approaches are not directly integrated into EOQ-type inventory frameworks with time-dependent demand.

2.6. Research Gap

Overall, the limitations identified in the literature are not confined to any single study but are inherent in a broader class of nonlinear EOQ-type inventory models. Existing approaches, including approximation-based and model-specific formulations, often lack generality, convergence guarantees, and robust numerical implementation frameworks. The present study addresses these methodological limitations by developing a convergence-guaranteed linearization-based optimization framework that is applicable to a wide range of nonlinear inventory systems with time-dependent demand and structural constraints. In this sense, the contribution of this work lies in advancing the general solution

methodology for nonlinear inventory optimization problems, rather than improving or correcting any specific existing model.

3. Materials and Methods

3.1. Formulation of the Wilson Model and Its Modifications

The Wilson inventory model, originally proposed in 1934, is one of the fundamental frameworks in inventory and supply chain optimization. It provides a mathematical structure for determining the optimal order quantity q_w and the optimal replenishment interval t_{sw}, which jointly minimize the total inventory cost over a finite planning horizon T. The model is based on the classical trade-off between ordering costs and holding costs under deterministic demand conditions.

Within this framework, the total cost function is defined as:

$$TC(q) = \frac{c_s \cdot D}{q} + \frac{1}{2} \cdot c_1 \cdot T \cdot q \rightarrow \min \quad (1)$$

The first term in (1) represents the total ordering (or transportation) cost, which decreases as the order quantity increases due to a reduction in ordering frequency. The second term represents the inventory holding cost, which increases linearly with the average inventory level. Thus, the optimization problem reflects a fundamental economic trade-off between operational efficiency and storage cost minimization.

The optimal solution of the classical Wilson model is derived by minimizing (1), resulting in the following expression:

$$q_w = \sqrt{\frac{2c_s \cdot D}{c_1 T}} \quad (2)$$

$$t_{sw} = \frac{q_w}{\mu} = \sqrt{\frac{2c_s \cdot D}{c_1 D}} \quad (3)$$

Where: c_s is the price of delivering goods to the market; D is the total demand for goods over time interval T; c1 is the cost of holding one unit of goods per day; μ is the daily demand. These expressions demonstrate that the optimal policy is governed by a square-root relationship between cost parameters and demand, reflecting the classical Economic Order Quantity (EOQ) structure.

Considering the critical remarks on the classical Wilson model provided by Slesarenko and Nestorenko (2014) and Nestorenko and Péliová, the basic supply management framework is further extended to incorporate more realistic economic assumptions. In particular, the revised formulation accounts for the time value of money and the financial returns generated from inventory investment. Within this generalized setting, the total cost function is expressed as follows Zeng et al., (2019):

$$TC(t_s) = (c_s + p \cdot \mu t_s)\frac{(1+r)^{t_s}((1+r)^T - 1)}{(1+r)^{t_s} - 1} - \frac{p\mu((1+r)^T - 1)}{ln(1+r)} \rightarrow \min \quad (4)$$

Here, p denotes the unit purchase price of the product, while r represents the expected daily rate of return (or discount factor) associated with capital invested in inventory. This formulation reflects a more advanced economic structure in which inventory decisions are influenced not only by physical holding costs but also by financial opportunity costs.

$$z = ln(1+r) \cdot t_s, \quad z_w = ln(1+r) \cdot t_{sw}, \quad Z = ln(1+r) \cdot T \quad (5)$$

This transformation converts multiplicative exponential expressions into additive forms, thereby allowing the model to be reformulated into a more tractable exponential structure. Under these notations, the total cost function (4) can be rewritten as follows:

$$TC(z) = \frac{p\mu(e^z-1)}{ln(1+r)}\left(\left(\frac{1}{2}z_w^2 + z\right)\frac{E^z}{E^z-1} - 1\right) \to min, \quad (6)$$

Where: $E^z = e^z$ denotes the exponential function. This representation clearly illustrates the nonlinear interaction between exponential discounting and quadratic cost accumulation effects, which jointly determine the optimal inventory policy.

The minimization of function (5) leads to the nonlinear first-order optimality condition:

$$E^z = 1 + z + \frac{1}{2}z_w^2. \quad (7)$$

This equation represents the equilibrium condition between marginal cost increases due to inventory holding and marginal cost reductions achieved through less frequent ordering. The presence of the exponential term reflects the effect of financial discounting, while the linear and quadratic terms capture the dynamics of inventory accumulation.

The solution of equation (7) defines the optimal replenishment structure under financial market conditions and generalizes the classical Wilson solution by incorporating time-dependent cost valuation effects.

An approximate solution of equation (7), obtained by applying a second-order Taylor series expansion with respect to the variable z, together with the notations introduced in (5), coincides with the classical Wilson optimal solution for the replenishment interval. In this case, the optimal time between orders is given by:

$$t_{so} = t_{sw} = \sqrt{\frac{2c_s}{rp\mu}}, \quad (8)$$

Consequently, the optimal number of orders within the planning horizon T in model (4)–(8), similarly to the classical Wilson framework, is determined as:

$$n_0 = \frac{T}{t_{s0}}$$

Models (1)–(2) and (4)–(8) are constructed under the assumption of equal inter-order times, that is, $\left(t_{si} = t_s, i = \overline{1,n}\right)$ where n denotes the total number of orders within the planning horizon T. Under the assumption of constant daily demand, the order quantities are also identical, i.e., $\left(q_i = q,\ i = \overline{1,n}\right)$.

In addition, Zeng et al. (2019) introduce a more general formulation that allows for a fixed number of orders n within the period T. In this case, the total cost function is defined as follows:

$$TC(t_{s1}, \dots, t_{sn}) = \sum_{i=1}^{n}(c_s + p \cdot \mu t_{si}) \times (1+r)^{T - \sum_{j=1}^{i-1} t_{sj}} \to min \quad (9)$$

or equivalently,

$$TC(z_1, \dots, z_n) = \frac{p\mu}{ln(1+r)}\sum_{i=1}^{n}\left(\frac{1}{2}z_w^2 + z_i\right)E^{Z - \sum_{j=1}^{i-1} z_j} \to min \quad (10)$$

The optimal solution $z_0 = (z_{01}, z_{02}, \dots, z_{0n})$ is obtained by solving a system of nonlinear equations. In Zeng et al. (2019), an approximate solution of this system is derived by employing series expansions of the functions E^z and $\sqrt{1+z}$, retaining terms up to and including the second order. This approximation enables the transformation of the original nonlinear optimization problem into a more tractable analytical form while preserving the essential structure of the optimal policy.

$$E\ z^i = \left(\frac{1}{2}\right)z_w^2 + 1 + z_{\{i+1\}}, i = 1, \dots, n-1$$

$$\sum(i = 1 \dots . n)z_i = Z \quad (11)$$

This results in three alternative representations of the optimal solution for the decision variable z_{oi} , which can be expressed in different equivalent analytical forms. In the first formulation, the optimal value is given by a correction of the average order size $\frac{z}{n}$ around the Wilson benchmark z_w, adjusted by a nonlinear term depending on the order index i, namely:

$$z_{oi} = z_w + \frac{\left(\frac{z}{n} - z_w\right)}{1+\frac{1}{2}z_w(n-1)}(1 + (1-i)z_w), \tag{12}$$

In the second representation, the optimal solution is expressed as a decomposition into the average allocation $\frac{z}{n}$ and a quadratic correction term that captures the deviation between the Wilson-based optimal level and the uniform distribution of demand across orders:

$$z_{oi} = \frac{z}{n} + \frac{1}{2}\left(\frac{1}{2}(n+1) - i\right)\left(z_w^2 - \frac{z^2}{n^2}\right), \tag{13}$$

The third formulation describes the optimal policy in an exponential form, where the allocation evolves geometrically along the order index i, reflecting compounding effects of the inventory dynamics:

$$z_{oi} = z_w + (z - nz_w)\frac{E^{z_w}-1}{E^{nz_w}-1} \cdot E^{(i-1)z_w} \tag{14}$$

Under these equivalent formulations, the minimum value of the total cost function can be expressed as:

$$TC_0 = \frac{p\cdot\mu}{ln(1+r)}\left(\left(\frac{1}{2}\cdot z_w^2 + 1 + z_{01}\right)\cdot E^Z - E^{z_{0n}}\right) \tag{15}$$

When the number of replenishment orders n over the planning horizon T is less than the optimal number of orders $n_0 = \frac{T}{t_{s0}}$, derived from model (4)–(8), the structure of the system implies that the intervals between successive orders increase over time, i.e., $t_{S01} < t_{S02} < \ldots t_{S01} < t_{S0n}$. Conversely, when $n \geq n_0$), these intervals exhibit a decreasing pattern.

Consequently, the optimal number of orders over the planning horizon T is determined by minimizing the total cost function (15), which yields the first-order optimality condition:

$$\frac{dTC_0(n)}{dn} = o$$

By incorporating the recursive relationship derived from (15), the following condition is obtained:

$$z_{0n} = z_w + (z_{01} - z_w)E^{(n-1)z_w}\ \frac{p\cdot\mu}{ln(1+r)} \cdot\left(\frac{dz_{01}(n)}{dn}\cdot E^{(n-1)z_w}(E^{z_w} - E^{z_{0n}}) - (z_{01} - z_w)\cdot z_w \cdot E^{(n-1)z_w + z_{0n}}\right) = 0 \tag{16}$$

It can be shown that Equation (16) is satisfied when:

$$z_{01} = z_{0n} = z_w$$

Hence, we obtain

$$z_{0i} = z_w, i = \overline{1, n_0} \tag{17}$$

and the fulfillment of equality

$$n_0 = \frac{T}{t_{sw}}, \tag{18}$$

where: t_{s_w}– is determined by formula (2.2)

Thus, in model (4)–(8), the restrictive assumption of equal time intervals between orders can be removed, since the minimum of the total cost function is achieved precisely when the inter-order times are constant, i.e., $\left(t_{si} = t_s, \text{for all } i = \overline{1,n}\right)$.

Furthermore, as shown in Konstantaras et al. (2019), it is possible to relax the assumption of a constant daily demand rate (μ = const) used in models (1) – (2) and (4) – (8), and instead assume a linear trend in demand:

$$\theta(t) = \mu + \frac{1}{2}\omega(t-1), \qquad t \in (1, T)$$

In this extended model, the condition$\left(t_{si} = t_s, i = \overline{1,n}\right)$, may still be assumed to hold, where n is the total number of orders within the planning horizon T.

Under these assumptions, the total cost function can be written as:

$$TC(t_s) = \sum_{i=1}^{n}\left(c_s + p\cdot\left(\left(\mu - \frac{1}{2}\omega\right)t_s + \frac{1}{2}\omega t_s^2\right) + (i-1)p\omega t_s^2)(1+r)^{(n-i+1)\cdot t_s} \quad (19)$$

Using the notations introduced above for z, zw , and Z, expression (19) can be rewritten as:

$TC(z) = \frac{p\mu(E^z-1)}{ln(1+r)}\left(\frac{1}{2}z_w^2 + \left(1-\frac{1}{2}\frac{\omega}{\mu}\right)z + \frac{1}{2}\frac{\omega}{\mu ln(1+r)}\cdot z^2\right)\cdot\frac{E^z}{E^z-1} - \frac{p\omega(E^z-1)}{\left(ln(1+r^2)\right)^2}\cdot z^2\cdot\frac{E^z}{(E^z-1)^2} + \frac{p\omega T}{ln(1+r)}\cdot z\cdot\frac{E^z}{E^z-1}.$ (20)

Using the Maclaurin series expansion of the function E^z and retaining only terms up to and including the second order in the variable z, the optimal solution of the minimization problem (20) can be obtained in the following form:

$$t_{s0} = \sqrt{\frac{\mu}{\mu+\frac{1}{2}\omega(T-1)}\cdot t_{sw}} \quad (21)$$

Considering expression (5) for t_{sw}, the optimal time between orders in the minimization problem (20), under the condition $\left(t_{si} = t_s,\ i = \overline{1,n}\right)\ t_s = t_{s0}$, is determined as t_{s0},

$$t_{s0} = \sqrt{\frac{2c_s}{rp\left(\mu+\frac{1}{2}\omega(T-1)\right)}}.$$

The optimal number of orders within the planning horizon T is then given by:

$$n_0 = \frac{T}{t_{s0}} = \frac{T}{\sqrt{\frac{2c_s}{rp\left(\mu+\frac{1}{2}\omega(T-1)\right)}}}.$$

and the optimal order quantity is expressed as:

$$q_{0i} = \left(\mu - \frac{1}{2}\omega\right)t_{s0} + \frac{1}{2}\omega t_{s0}^2 + (i-1)p\omega\ t_{s0}^2, i = \overline{1,n} \quad (22)$$

Alongside model (19) – (22), Zeng et al. (2019) consider a modified formulation with a constant order size over the planning horizon T. In this case, the total cost function is defined as follows:

$$TC(q) = (c_s + pq)\sum_{i=0}^{n-1}(1+r)^{T-T_i} \quad (23)$$

where the replenishment epochs Ti are given by:

$$T_i = \frac{\sqrt{\left(\mu-\frac{1}{2}\omega\right)^2 + 2i\omega q} - \left(\mu-\frac{1}{2}\omega\right)}{\omega} \quad (24)$$

After appropriate transformations, model (23)–(24) can be rewritten in the following form:

$$TC(z) = \frac{p\cdot\left(\mu-\frac{1}{2}\omega\right)}{ln(1+r)}\left(\frac{1}{2}\cdot z_w^2 + z\right)\cdot\ \sum_{i=0}^{n-1}E^{Z-Z_i} \quad (25)$$

where:

$$Z_i = \frac{\sqrt{1+2i\theta z}-1}{\theta}, \quad \theta = \frac{\omega}{ln(1+r)\left(\mu-\frac{1}{2}\omega\right)} \quad (26)$$

Using the Maclaurin series expansions of the functions E^z and $\sqrt{1+z}$, and retaining terms up to and including the second order, the following approximate optimal solutions of the minimization problem (23) under the constraint $q_i = q, i = \overline{1,n}$ are obtained:

Similarly, Zeng et al. (2019), using Maclaurin series expansions and considering terms up to the second order, derived the following approximate expressions for the optimal solution of the minimization of the cost function (23) under the same constraint:

$$q_0 = \sqrt{\frac{\mu+\frac{1}{2}\omega(T-1)}{\mu}\cdot q_w}, \qquad q_0 = \sqrt{\frac{2c_s\left(\mu+\frac{1}{2}\omega(T-1)\right)}{rp}}. \tag{27-28}$$

In this case, the optimal time between orders is determined as follows:

$$t_{s0i} = \frac{1}{\omega}\left(\sqrt{\left(\mu-\frac{1}{2}\omega\right)^2+2iwq_0}-\sqrt{\left(\mu-\frac{1}{2}\omega\right)^2+2(i-1)\omega q_0},\right) i=\overline{1,n} \tag{29}$$

and the optimal number of orders over the planning horizon T is given by the formula:

$$n_0 = \frac{D}{q_0} = \frac{\left(\mu+\frac{1}{2}\omega(T-1)\right)\cdot T}{\sqrt{\frac{2cs\left(\left(\mu+\frac{1}{2}\omega(T-1)\right)\right)}{rp}}}.$$

Finally, for the case of linearly time-dependent demand and a fixed number of orders within the planning horizon T, a further extension of the total cost function is introduced in Zeng et al. (2019) as follows:

$$TC(t_1,t_2,\dots,t_n) = \sum_{i=1}^{n}\left(c_s+p\cdot\left(\mu-\frac{1}{2}\omega+\omega(t_1+t_2,\dots,+t_{i-1})+\frac{1}{2}\omega t_i\right)\cdot t_i\right) \times (1+r)^{T-(t_1+t_2+\dots+t_{i-1})} \tag{30}$$

After transformation, the model takes the following form:

$$TC(z_1,z_2,\dots,z_n) = \frac{p\left(\mu-\frac{1}{2}\omega\right)}{ln(1+r)}\times\sum_{i=1}^{n}\left(\frac{1}{2}z_w^2+\left(1+\theta(z_1+z_2+\dots+z_{i-1})+\frac{1}{2}\theta z_i\right)\cdot z_i\right) \cdot E^{Z-(z_1+z_2+\dots+z_{i-1})} \tag{31}$$

Where:

$$\theta = \frac{\omega}{ln(1+r)\left(\mu-\frac{1}{2}\omega\right)}.$$

It is stated in Zeng et al. (2019) that the minimum of model (30) is achieved at the point $z_0 = (z_{01}, z_{02}, \dots, z_{0n})$, which is the solution of the nonlinear system:

$$\begin{gathered}\left(1+\theta(z_1+z_2+\dots+z_i)\right)(e^{z_i}-1) = \left(\frac{1}{2}\right)w\,z_i^2+z_i, i=1,\dots,n-1\\ \left(1+\theta(z_1+z_2+\dots+z_i)\right)z_{i+1}+\left(\frac{1}{2}\right)\theta\,z_{i+1}^2, i=1,\dots,n-1\\ \textstyle\sum_{i=1}^{n} z_i = Z\end{gathered} \tag{32}$$

The solution of system (32) can be obtained using standard iterative methods of nonlinear programming. Under constant demand conditions μ, the cost function attains its minimum when conditions (17)–(18) are satisfied. Based on this observation, for the linear demand trend case in model (30)–(32), the following hypotheses are introduced:

1. The optimal number of orders over the planning horizon *T* is given by:

$$n_0 = \frac{D}{q_0} = \frac{\left(\mu+\frac{1}{2}\omega(T-1)\right)\cdot T}{\sqrt{\frac{2c_S\left(\mu+\frac{1}{2}\omega(T-1)\right)}{rp}}} = \frac{T}{\sqrt{\frac{2c_S}{rp\left(\mu+\frac{1}{2}\omega(T-1)\right)}}} = \frac{T}{t_{s0}} \quad (33)$$

2. The minimum of the total cost function coincides with the optimal solutions obtained in models (23)–(29) and (19)–(22).

These assumptions enable, in practical applications with linear demand trends, the use of model (30)–(32) under the simplifying condition that either equal time intervals between orders or equal order quantities are maintained, with the number of orders satisfying relation (33).

Assuming that the first hypothesis holds, the subsequent section verifies the second hypothesis for model (30) by employing the linearization method developed by B. N. Pshenichny (1975, 1983) for solving nonlinear programming problems. This approach enables the construction of a numerical algorithm for solving modified Wilson EOQ models in retail food supply systems, ensuring convergence in a finite number of iterations.

3.2 Linearization Method for Solving a Nonlinear Programming Problem with Equality Constraints

Compared to alternative nonlinear optimization techniques such as Newton-type methods, Sequential Quadratic Programming (SQP), and interior-point algorithms, the Pshenichny linearization approach offers a structurally simpler iterative framework. While Newton and SQP methods generally require second-order derivative information and may involve higher computational complexity per iteration, the linearization-based method transforms the original nonlinear problem into a sequence of auxiliary quadratic programming subproblems with more stable and tractable computational properties. In the context of large-scale or structurally constrained EOQ-type models, this feature can improve numerical stability and reduce implementation complexity, particularly when the objective function exhibits nonlinear but smooth behavior.

The nonlinear programming problem with equality constraints can be formulated as follows:

$$f_0(x) \to minx = (x^1, \ldots, x^n) \in R^n, \quad (34)$$

subject to the constraints:

$$f_0(x) = 0, i = 1, \ldots, m \quad (35)$$

In the case of linear equality constraints, system (34) can be represented in matrix form as:

$$Ax = b, \quad (36)$$

where: A is a matrix of coefficients of size m × n, b – column vector of size m×1

In the linearization method developed by Pshenichny (1983), two auxiliary constructs are introduced. First, the maximal constraint violation function is defined as follows:

$$F(x) = max|f_i(x)|, \quad I = \{i|i = 1, \ldots, m\}$$

and the index set of active constraints is defined as:

$$I_\delta(x) = \{i \in I : f_i(x) \geq F(x) - \delta\}, \qquad \delta > 0$$

Based on these definitions, an auxiliary quadratic programming subproblem is constructed as follows:

$$\min_p \left\{ \left(f'_0(x), p\right) + \frac{1}{2}\|p\|^2 : \left(f'_i(x), p\right) + f_i(x) = 0, \quad i \notin I_\delta(x) \right., \quad (37)$$

where r∈|Rn is the search direction vector, $\|p\| = \sum_{i=1}^{n}\left(p^i\right)^2$ is the Euclidean norm, and pi are its components.

The gradient vectors are defined as:

$$f'_0(x) = grad f_0(x) = \left(\frac{\partial f_0}{\partial x^1}, \dots, \frac{\partial f_0}{\partial x^n}\right),$$

$$f'_i(x) = grad f_i(x) = \left(\frac{\partial f_i}{\partial x^1}, \dots, \frac{\partial f_i}{\partial x^n}\right)$$

The inner product of two vectors a, b ∈ Rn is given by:

$$(a, b) = \sum_{i=1}^{n} a_i, b_i,$$

a_i, b_i- components of the vectors a and b.

The following assumptions are imposed for the validity and convergence of the linearization procedure:

a) There exists a bounded level set:

$$\Omega_N = \{x : f_0(x) + NF(x) \le C_0, C_0 = f(x_0) + NF(x_0)$$

where: $x_0 \in R^n$ is an initial approximation of the solution, and N>0 is a sufficiently large constant.

b) The gradients of the constraint functions satisfy the Lipschitz condition in Ω_N :

$$\left\|f'_i(x_1) - f'_i(x_2)\right\| \le L \cdot \|x_1 - x_2\|$$

c) The auxiliary quadratic programming problem (36) is solvable for any x∈Ω_N, and there exist Lagrange multipliers $u^i(x)$

$$\sum_{i \notin I} u^i(x) \le N, u^i \in 0,$$

These multipliers are associated with the Lagrange function:

$$L(x, u) = f_0(x) + \sum_{i=1}^{m} u^i f^i(x) \qquad (38)$$

When selecting a numerical method for solving problem (37), it is important to note that this problem is auxiliary in nature; therefore, its solution must be obtained within a finite number of computational steps. Moreover, since the constant N defining the set Ω_N, is generally unknown a priori, it is desirable to monitor the correctness of its choice during the iterative process. For this purpose, the computation of the corresponding Lagrange multipliers $u^i(x)$. becomes particularly useful. Consequently, it is reasonable to reformulate the auxiliary problem in its dual form.

The nonlinear programming problem with equality constraints can be formulated as follows:

$$\min_p\{(p, C_p) + (d, p)\} \qquad (39)$$

subject to:

$$A_p = b$$

where:

$C = I_n, d = f_0(x), b = -f(x), \quad A = f'(x)$ I_n is the identity matrix $f(x) \in R^{m(x)}, m(x) = |I_\delta(x)|$ the number of elements in $I_\delta(x)$, $f'(x)$is the matrix with rows $f_i(x)$, $i \in I_\delta(x)$.

Then, according to Pshenichny (1983) and Pshenichny and Danilin (1975), the dual problem corresponding to problem (39) consists of maximizing the following quadratic function:

$$\max_u \left\{\frac{1}{2}(C^{-1}[d + A^*u], d + A^*u) - (b, u)\right\} \quad (40)$$

subject to the constraints:

$$u^i \geq 0,\ i \in I_\delta(x) \quad (41)$$

If the rows $a_i\ A_p = b$ are linearly independent, then the solution of the dual problem is represented by the formula Pshenichny (1983):

$$u = -(AC^{-1}A^*)^{-1}[b + AC^{-1}d] \quad (42)$$

and the solution of the primal problem (36) is determined as:

$$p = -C^{-1}[d + A^*u] \quad (43)$$

Considering the expressions for C and A, formulas (42) and (43) can be rewritten in the following form:

$$u(x) = -\left(f'(x)\left(f'(x)\right)^*\right)^{-1}\left[-f(x) + f'(x)\left(f'_0(x)\right)^*\right] \quad (44)$$

$$p(x) = -\left[\left(f'_0(x)\right)^* + \left(f'(x)\right)^* u(x)\right] \quad (45)$$

The vector u(x) represents the Lagrange multipliers associated with the active constraints, while p(x) defines the search direction used in the iterative minimization process. The proposed linearization algorithm combines quadratic approximation, dual optimization, and adaptive penalty correction. Its main advantage lies in the finite-step solution of the auxiliary subproblems and the guaranteed decrease of the generalized penalty functional, which makes the method effective for solving modified Wilson EOQ optimization models with nonlinear demand structures.

Algorithm (of the minimization method for problem (34), (36)).

We choose an arbitrary initial approximation x0, the accuracy ε1 of the solution of the original problem (34) - (36), the accuracy ε2 of the auxiliary problem (37) and the quantities $N_0 > 0, 0 < \varepsilon < 1,\ \delta_0 = F(x_0) + \varepsilon,$ We will get that the rows ai of the matrix A are less independent. Let the point xk and the numbers Nk, δk are given.

The general step of the algorithm consists of the following points

1. We construct $F(x_k)$ and the set $I_\delta(x_k)$ c $m(x_k) = |I_\partial(x_k)|$
2. Let us solve the dual problem (40)-(41), its solution is the Lagrange multipliers $u_k^i, i \in I_\partial(x_k)$, determined by formulas (44). If the problem (40)-(41) is incompatible, then we assume$x_{k+1} = x_k, \delta_{k+1} = \frac{\delta_k}{2} N_{k+1} = N_k$ and return to n.1.
3. If a solution to problem (40)-(41) is obtained and the inequality $N_{k+1} > \sum_{i \in I_\partial(x_k)} |u_k^i|$ is satisfied, then, we assume $N_{k+1} = N_k$. Otherwise, $N_{k+1} > 2\sum_{i \notin I_\partial(x_k)} |u_k^i|$.
4. Calculate the solution of the auxiliary problem (37) using formula (45).
5. Calculate $\|p_k\|^2 = \sum_{j=1}^n \left(p_k^{(j)}\right)^2$ if $\|p_k\|^2 \leq \varepsilon_1$, then go to 6. Other wise we assume $x_{k+1} = x_k + \alpha_k * p_k, \delta_{k+1} = \delta_k$, where the step α_k is chosen equal to $\frac{1}{2^{q_0}}$ and q_0 is the first of the integers $q = 0,1\ldots$, for which the relation $\phi_{N_{k+1}} \quad \phi_{N_{k+1}}\left(x_k + \frac{1}{2^q} p_k\right) \leq \phi_{N_{k+1}}\left(x_k - \frac{1}{2^q}\varepsilon\|p_k\|^2\right)$,
   where $\phi_N(x) = f_0(x) + N * F(x)$
6. Exit.

The proposed linearization algorithm combines quadratic approximation, dual optimization, and adaptive penalty correction. Its main advantage lies in the finite-step solution of the auxiliary subproblems and the guaranteed decrease of the generalized penalty functional, which makes the method effective for solving modified Wilson EOQ optimization models with nonlinear demand structures.

The convergence guarantee of the proposed method holds under standard assumptions of the Pshenichny linearization framework, including differentiability of the objective function, boundedness of the feasible region, and satisfaction of regularity conditions for the iterative scheme. In the present EOQ formulation, these conditions are satisfied due to the smooth structure of the cost function and the convexity properties of the quadratic subproblems generated at each iteration. Consequently, the iterative procedure is expected to converge to a stationary point of the original nonlinear optimization problem, provided that the initial solution lies within a feasible neighborhood of the optimal region.

3.3 Solution of Nonlinear Programming Problems for Modified Wilson EOQ Models with a Linear Demand Trend Using the Linearization Method

Let us consider the nonlinear optimization problem associated with the minimization of the total cost function defined by model (30), under the assumption that the intervals between successive orders form a linear sequence over the planning horizon T.

$$\begin{cases} t_i = t_1 + (i-1)t_s \ \left(i = \overline{1,n}\right) \\ \sum_{i=1}^{n} t_i = T \end{cases} \tag{46}$$

Where: the variables $t_1, \dots, t_n$ and the parameter t_s are to be determined from the optimization problem

$$TC(t_1, \dots, t_{\tilde{n}}) \to \min \tag{47}$$

under constraints (46). Here, the objective function formally includes an additional auxiliary variable $t_{\tilde{n}}$, where:

$$\tilde{n} = n + 1$$

and the coefficient corresponding to this variable is assumed to be zero.

Introducing the notation:

$$t_{n+1} = \tilde{t}_s, \quad \tilde{t}_s = -t_s$$

constraints (46) can be rewritten in equivalent form as:

$$\begin{cases} t_i = t_1 + (i-1)t_s \ \left(i = \overline{1,n}\right) \\ \sum_{i=1}^{n} t_i = T \end{cases} \tag{48}$$

or, equivalently, as a system of linear equality constraints for the vector

$$t = (t_1, \dots, t_{n+1})$$

in the form:

$$\begin{cases} f_i(t) \equiv t_1 - t_i - (i-1) \cdot t_{n+1} = 0 \ \left(i = \overline{1,n}\right) \\ f_{i+1}(t) \equiv \sum_{i=1}^{n} t_i - T = 0 \end{cases} \tag{49}$$

Introducing the notation

$$x^i = t_i (i = 1, \dots, \tilde{n} \ \ \tilde{n} = n+1),$$
$$x = \left(x^1, \dots, x^{\tilde{n}}\right) \in R^{\tilde{n}}$$

problem (46)–(48) is reduced to the general nonlinear programming problem (33)–(34) with the objective function:

$$f_0(x) = \sum_{i=1}^{n} \left(c_s + p\left(\mu - \frac{1}{2}\omega + \omega(x^1 + x^2 + \dots + x^{i-1}) + \frac{1}{2}\omega x^i\right) \cdot x^i\right) \cdot (1 + r)^{T - \left(x^1 + x^2 + \dots + x^{i-1}\right)} \tag{50}$$

subject to the equality constraints:

$$\begin{cases} f_i(x) \equiv x^i - x^1 + (i-1) \cdot x^{\tilde{n}} = 0, \ i = 1, \dots, n \\ f_{\tilde{n}}(x) = \sum_{i=1}^{n} x^i - T = 0, \ \ \tilde{n} = n+1, \end{cases} \tag{51}$$

Where

$$f(x) \equiv \left(f_1(x), \ldots, f_m(x)\right), \qquad f'_i(x) = \left(\frac{\partial f_i(x)}{\partial x^1}, \ldots, \frac{\partial f_i(x)}{\partial x^{\tilde{n}}}\right) \left(i = \overline{1,m} \;\; m = \tilde{n}\right)$$

The derivatives of the nonlinear objective function $f_0(x)$ are obtained analytically.

For the first variable x1, the derivative takes the form:

$$\frac{\partial f_0(x)}{\partial x^1} = \left(\frac{1}{2}p\omega \cdot x^1 + (c_s + p\left(\mu - \frac{1}{2}\omega + \omega x^1 + \frac{1}{2}\omega \cdot x^1\right)\right)(1+r)^T + p\omega \cdot x^2 \times$$

$$\times (1+r)^{T-x^1} + \{c_s + p\left(\mu - \frac{1}{2}\omega + \omega x^1 + \frac{1}{2}\omega x^2\right) \cdot x^2\}(-1)(1+r)^{T-x^1} ln(1+r) +$$

$$+ \sum_{i=3}^{n} p\omega \cdot x^i (1+r)^{T-\left(x^1 - x^2 + \ldots + x^{i-1}\right)} + \left\{c_s + p\left[\mu - \frac{1}{2}\omega\left(x^1 + x^2 + \ldots + x^{i-1}\right) + \right.\right.$$

$$\left.\left. + \frac{1}{2}\omega x^i\right] \cdot x^i\right\} \cdot (-1) \cdot (1+r)^{T-\left(x^1 + x^2 + \ldots + x^{i-1}\right)} \times ln(1+r)$$

Similarly, for variables xj, j=2,,n, the derivatives are written as:

$$\frac{\partial f_0(x)}{\partial x^j} = \left(\frac{1}{2}p\omega \cdot x^j + (c_s + p(\mu - \frac{1}{2}\omega + \omega\left(x^1 + x^2 + \cdots x^{j-1}\right) + \frac{1}{2}\omega x^j\right)\right) \times (1+r)^{T-\left(x^1 + x^2 + \cdots x^{i-1}\right)}$$
$$+ \sum_{i=j+1}^{n} (p\omega \cdot x^1) \times (1+r)^{T-\left(x^1 + x^2 + \cdots x^{i-1}\right)}$$
$$+ \left\{c_s p\left[\mu - \frac{1}{2}\omega + + \omega\left(x^1 + x^2 + \cdots x^{i-1}\right) + \frac{1}{2}\omega x^i\right] \cdot x^i\right\} \times$$
$$\times (-1)(1+r)^{T-\left(x^1 + x^2 + \cdots x^{i-1}\right)} ln(1+r) \;\; (j = 2, \ldots, n), \frac{\partial f_0(x)}{\partial x^{\tilde{n}}} = 0 \qquad (52)$$

For the auxiliary variable $x^{\tilde{n}}$, the derivative is equal to zero:

The derivatives of the constraint functions $f_i(x)$ are determined directly from equations (51):

$$\frac{\partial f_i(x)}{\partial x^j} = \begin{cases} 0, \;\; if \;\; j \le \tilde{n}, \;\; i = 1 \\ -1, \;\; if \;\; j = 1, \;\; 1 < i \le n \\ 1, \;\; if \;\; j + \tilde{n}, \;\; j = i, \;\; 1 < i \le n \\ 0, \;\; if \;\; j = n, \;\; j + i, \;\; 1 < i \le n \\ 1, \;\; if \;\; j \le n, \;\; i = n + 1 \\ 0, \;\; if \;\; j = n + 1, \;\; i = n + 1 \end{cases} \qquad (53)$$

For the original nonlinear optimization problem

$$\min_x \{f_0(x) \quad f_i(x) = 0 \qquad i = \overline{1, m(\tilde{x})}, \qquad \tilde{x} \in R^{\tilde{n}} \qquad (54)$$

with functions defined by formulas (51) – (52), the corresponding auxiliary quadratic programming problem (39) is characterized by the constraint matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & 1 & \cdots & 0 & 1 & n-2 \\ -1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & n-1 \\ 1 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & n \end{pmatrix} \quad (55)$$

Model (19), corresponding to a linear trend in successive orders under the condition of equal time intervals between replenishments, can be formulated as the following nonlinear programming problem with respect to the variables:

$$t = (t_{s1}, \dots, t_{s,n+1}), \quad t_{s,n+1} = -t_s, \quad \tilde{n} = n+1$$

The total cost function is written as:

$$TC(t_{s1}, t_{s2}, \dots, t_{s,n+1}) = \sum_{i=1}^{n} (c_s + p \times \left(\mu - \frac{1}{2}\omega\right). t_{s,n+1} + \frac{1}{2}\omega t_{s,n+1}^2) + (i-1)p\omega t_{s,n+1}^2)$$
$$+(1+r)^{-(n-i+1)t_{s,n+1}} \to \min \quad (56)$$

subject to the constraints:

$$\begin{cases} f_i(t) = t_i - t_1 - (i-1)t_{s,n+1} = 0 \quad i = \overline{1,n}, \\ \sum_{i=1}^{n} t_i = T \end{cases} \quad (57)$$

Similarly, model (24), corresponding to a linear trend in successive orders under the condition of equal order quantities, can be represented as a nonlinear programming problem with respect to the variables

$$(q_1, \dots, q_{n+1}), \qquad q_{n+1} = -q, \ \tilde{n} = n+1$$

The objective function becomes:

$$TC(q_1, \dots, q_{n+1}) = (c_s + pq_{n+1}) \sum_{i=0}^{n-1} (1+r)^{T-T_i} \to min \quad (58)$$

where

$$T_i = \frac{1}{\omega}\left(\sqrt{\left(\mu - \frac{1}{2}\omega\right)^2 + 2 \cdot i \cdot \omega \cdot q_{n+1}} - \left(\mu - \frac{1}{2}\omega\right)\right)$$

The equality constraints are given by:

$$\begin{cases} f_i(q_1, \dots, q_{n+1}) \equiv q_i + q_{1i} = 0, \ i = \overline{1,n}, \\ f_{\tilde{n}}(q_1, \dots, q_{n+1}) \equiv \mu + \frac{1}{2}\omega(T-1)T + n\, q_{\tilde{n}} = 0 \end{cases} \quad (59)$$

The nonlinear programming problems with linear equality constraints (56) – (57) and (58) –(59) are solved using the same linearization algorithm developed for problem (47)–(48).

Reformulation of Problem (56) – (57)

$$x \equiv (x^1, \dots, x^{\tilde{n}}), \qquad \tilde{n} = n+1$$

with

$$x_i = t_{si} \qquad i = \overline{1,n},$$

the objective function takes the form:

$$f_0(x) = \sum_{i=1}^{\tilde{n}} \left(c_s + p\left(\mu - \frac{1}{2}\omega\right)x^{\tilde{n}} + \frac{1}{2}\omega \cdot \left(x^{\tilde{n}}\right)^2\right) + (i-1)p\omega\left(x^{\tilde{n}}\right)^2 \cdot (1+r)^{-(n-i+1)x^{\tilde{n}}}$$
$$\to min \quad (60)$$

$$\begin{cases} f_i(x) \equiv x^i - x^1 - (i-1)x^{n+1} = 0, \ \left(i = \overline{1,n}\right) \\ f_{\tilde{n}}(x) \equiv \sum_{i=1}^{\tilde{n}} x^i - T = 0, \end{cases} \tag{61}$$

subject to

$$\frac{\partial f_0(x)}{\partial x^j} = 0 \quad j = \overline{1,n};\ \frac{\partial f_0(x)}{\partial x^{\tilde{n}}}$$
$$= \sum_{i=1}^{n} \left\{ \left[p\left(\mu - \frac{1}{2}\omega\right) + \omega \cdot x^{\tilde{n}} + 2(i-1)p\omega x^{\tilde{n}} \right] \cdot (1+r)^{-(n-i+1)x^{\tilde{n}}} \right.$$
$$+ \left[c_s + p\left(\mu - \frac{1}{2}\omega\right) x^n + \frac{1}{2}\omega \right.$$
$$\left. \left. \cdot \left(x^{\tilde{n}}\right)^2 + (i-1)p\omega\left(x^{\tilde{n}}\right)^2 \right] \left(-(n-i+1) \times (1+r)^{-(\tilde{n}-i+1)x^{\tilde{n}}} \cdot ln(1+r)\right) \right\} \tag{62}$$

Reformulation of Problem (57) – (58)

Using the notation

$$x \equiv \left(x^1, \ldots, x^{\tilde{n}}\right), \quad \tilde{n} = n+1, x^i = q_i \left(i = \overline{1,n}\right), \ x^{\tilde{n}} = q_{n+1}$$

the objective function becomes:

$$f_0(x) = \left(c_s + px^{\tilde{n}}\right) \sum_{i=0}^{n-1} (1+r)^{T-T_i} \to min$$

where

$$T_i = \frac{1}{\omega}\left(\sqrt{\left(\mu - \frac{1}{2}\omega\right)^2 + 2 \cdot i \cdot \omega \cdot x^{\tilde{n}}} - \left(\mu - \frac{1}{2}\omega\right) \right) \tag{63}$$

The equality constraints are written as:

$$\begin{cases} f_i(x) \equiv x^i + x^{\tilde{n}} = 0, \quad i = \overline{1,n}, \\ f_{\tilde{n}}(x) \equiv \mu + \frac{1}{2}\omega(T\ -1)T + nx^{\tilde{n}} = 0 \end{cases}$$
(63)

The derivatives of the objective function satisfy:

$$\frac{\partial f_0(x)}{\partial x^j} = 0, \qquad j = \overline{1,n},$$

and

$$\frac{\partial f_0(x)}{\partial x^{\tilde{n}}} = p \sum_{i=0}^{\tilde{n}-1} (1+r)^{T-T_i}$$
$$+ \left(c_s + px^{\tilde{n}}\right)$$
$$\times \left[- ln\,(1+\varepsilon) \sum_{i=0}^{n-1} \frac{i}{\sqrt{\left(\mu - \frac{1}{2}\omega\right)^2 + 2i\omega x^{\tilde{n}} \cdot}} (1+r)^{T-T_i} \right] \tag{65}$$

Derivatives of the Constraint Functions

The derivatives

$$\frac{\partial f_i(x)}{\partial x^j} = 0 \qquad \left(j = \overline{1,\tilde{n}}\right)$$

for problems (56)–(57) and (58)–(59) are calculated analogously to problem (47)–(48) using formulas (53).

$$x^i = q_i \left(i = \overline{1,n}\right), \quad x^{\tilde{n}} = -q$$

the derivatives are determined as:

$$\frac{\partial f_i(x)}{\partial x^j} = 1\left(i = \overline{1,n}\,, \qquad j = \overline{1,\tilde{n}}\right)$$

$$\frac{\partial f_{\tilde{n}}(x)}{\partial x^j} = \begin{cases} 0, & j < \tilde{n} \\ 1, & j = \tilde{n} \end{cases}$$

Thus, both modified Wilson EOQ models with linear demand trends and equality constraints can be reformulated as nonlinear programming problems with linear equality constraints and efficiently solved using the finite-step linearization procedure proposed by B. N. Pshenichny.

4. Numerical implementation of the method for the modified EOQ – Wilson models (50)-(51) and discussion of the results

To illustrate the behavior of the proposed linearization approach and to demonstrate the computational procedure of the modified Wilson EOQ model with a linear demand trend, an illustrative numerical example is considered for problem (50)–(51).

The numerical implementation is carried out using the iterative algorithm developed in the previous sections, based on the linearization method of B. N. Pshenichny. The parameter values are adopted from Zeng et al. (2019), which ensures consistency and enables a direct comparison with previously reported results.The following input parameters were considered:

- C_s=400 conventional monetary units;
- p = 20 conventional monetary units;
- r = 0.001;
- μ = 25 units/day;
- ω = 0.066;
- T = 333 days.

Using formulas (21), (28), and (33), the preliminary approximate value of the optimal number of orders n0 was first determined. After that, the nonlinear optimization problem (50)–(51) was solved numerically for three neighboring variants of the order frequency:

The computational results are illustrated in Figures 1–3, respectively.

The numerical implementation demonstrates that the proposed linearization method converges steadily for all considered values of n. The iterative process required only a limited number of iterations to satisfy the stopping criterion, confirming the computational stability of the auxiliary quadratic programming procedure and the effectiveness of the adaptive penalty parameter correction.

The obtained solutions show that the total cost function is highly sensitive to changes in the number of replenishment cycles over the planning horizon. In particular, the minimum value of the objective function is achieved in the vicinity of $n = n_0$, which experimentally confirms the validity of the analytical approximation proposed in Zeng et al. (2019),

At the same time, the cases $n = n_0 - 1$ and $n = n_0 + 1$ produce slightly higher total costs, indicating that even a small deviation from the optimal replenishment frequency leads to measurable increases in inventory and ordering expenses. This observation confirms the economic significance of accurately determining the optimal order schedule in retail food logistics systems characterized by linearly changing demand.

The numerical experiments also demonstrate that the assumption regarding equal or nearly equal time intervals between successive replenishments remains sufficiently accurate for practical applications. Therefore, the simplified analytical estimates derived from the modified Wilson EOQ framework can be effectively used as initial approximations for the nonlinear optimization algorithm.

An additional advantage of the proposed method is that it avoids direct solution of the original nonlinear system (32) by replacing it with a sequence of auxiliary quadratic programming problems possessing explicit analytical structure. This significantly reduces computational complexity and improves numerical tractability for large-scale inventory planning problems.

Overall, the numerical study demonstrates that the proposed linearization procedure is capable of generating stable and feasible solutions for the considered EOQ model with time-dependent demand. The obtained results suggest that the approach may serve as a useful tool for inventory optimization under deterministic demand trends. Nevertheless, further computational investigations involving additional parameter settings and larger problem instances are required to comprehensively assess its numerical robustness and broader applicability.

In contrast to the approximate optimal solution of this problem obtained in Zeng et al. (2019), where the functions ez and $\sqrt{1+z}$ were expanded into power series and only terms up to the second order $z^k(k \leq 2)$ were retained without a rigorous estimation of the approximation error, in the present study we derive the solution using a linearization method Pshenichny (1983). This approach provides controlled accuracy levels of $\varepsilon_1 = 10^{-2}$ and $\varepsilon_2 = 10^{-3}$ for solving the auxiliary quadratic programming problem (36).

For the solution of problem (36), we employ its dual formulation (39) as a quadratic programming problem in terms of the Lagrange multipliers u^i, subject to the non-negativity constraints $u^i \geq 0$

Problem (49)–(50) is formulated as a nonlinear programming problem with equality constraints under the assumption of equal time intervals t_s , between successive orders. In this formulation, the inter-order time t_s, unlike in Zeng et al. (2019), is treated as an additional decision variable t_{n+1}, augmenting the primary variables $t_1, \ldots, t_n$ where n denotes the total number of orders within the planning horizon T.

As expected, when the number of orders equals the classical EOQ/Wilson value $n = n_0$ and the condition of equal time intervals holds, the optimal solution satisfies equal order quantities (Fig. 1). For $n < n_0$, order quantities increase over time (Fig. 2), whereas for $n > n_0$, they decrease monotonically (Fig. 3).

This structural property of order quantities under equal inter-order times in the modified Wilson EOQ model (29) with a linear demand trend was originally hypothesized in Zeng et al. (2019), and is confirmed by our numerical results obtained via the linearization method.

Software implementation. The proposed algorithm was implemented in Borland C++ using a custom-developed computational code. All figures presented in the paper were generated from the outputs of this implementation.

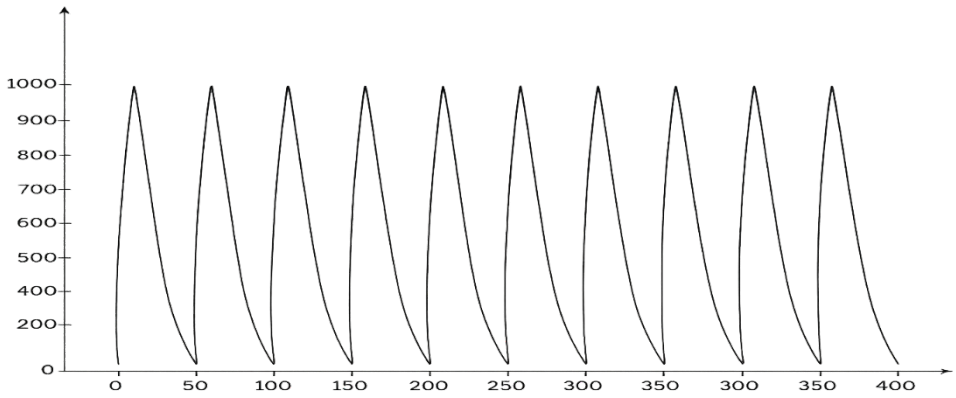


Fig.1: Dynamics of the inventory level q under a balanced replenishment configuration

$(n = n_0)$, which serves as a reference case for comparison with alternative replenishment strategies over the planning horizon T. This configuration represents the baseline structure of the system.

Figure 1 illustrates the dynamics of the inventory level qqq under a balanced replenishment configuration (n=n0) over the planning horizon T. This configuration is considered the baseline (reference) scenario and serves as a benchmark for comparing the performance of alternative replenishment strategies.

As shown in the figure, the inventory level rises sharply to its maximum value at the beginning of each replenishment cycle and then gradually decreases over time. This decline occurs as customer demand is satisfied from the available stock, causing the inventory level to approach a minimum value just before the next replenishment event. At the end of each cycle, a new replenishment order is received, restoring the inventory level to a high value and initiating the next cycle.

The periodic “sawtooth” pattern observed in the graph is characteristic of inventory management systems operating under regular replenishment policies. In the balanced configuration, both the replenishment quantity and the replenishment interval are consistently maintained, resulting in a stable and repetitive inventory trajectory throughout the planning horizon. This behavior helps prevent stockouts while also limiting the costs associated with holding excessive inventory.

Figure 1 represents the standard operating mode of the system and provides a reliable reference framework for evaluating the impact of alternative replenishment policies on inventory levels, replenishment frequency, and overall operational efficiency.

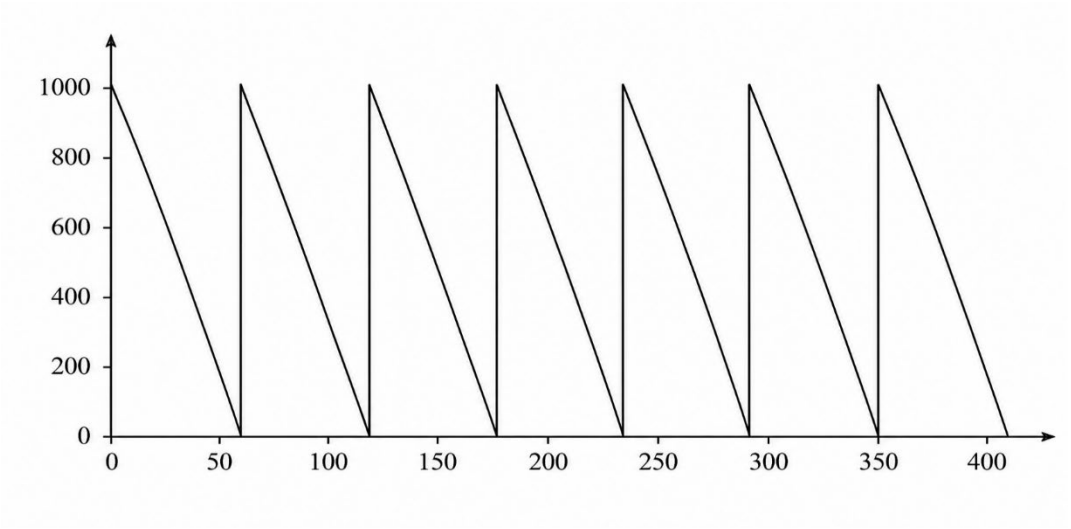


Fig.2: Dynamics of the inventory level q under a reduced number of replenishment cycles $(n < n_0)$.

In this case, fewer replenishments lead to higher order quantities and longer replenishment intervals over the planning horizon *T*, resulting in more pronounced inventory fluctuations.

Figure 2 illustrates the dynamics of the inventory level qqq when the number of replenishment cycles is reduced $n < n_0$. Compared with the balanced replenishment configuration, fewer replenishment events occur during the planning horizon T, which results in larger order quantities and longer intervals between successive replenishments.

The inventory level exhibits a characteristic sawtooth pattern, where stock is replenished instantaneously to its maximum level and then decreases linearly as demand is satisfied over time. Because replenishments are less frequent, each cycle covers a longer period, causing inventory to remain at higher levels immediately after replenishment and decline steadily until reaching zero before the next order arrives.

This policy leads to more pronounced inventory fluctuations, as evidenced by the wider spacing between replenishment points and the larger inventory swings within each cycle. While reducing the number of replenishment orders may lower ordering and administrative costs, it also increases the average inventory held in stock, potentially leading to higher holding costs. Consequently, the system faces a trade-off between replenishment frequency and inventory carrying costs.

Figure 2 demonstrates how a reduced replenishment frequency affects inventory behavior by increasing cycle lengths and order quantities, creating larger variations in inventory levels throughout the planning horizon. This scenario provides a useful basis for evaluating the operational and economic implications of less frequent replenishment strategies.

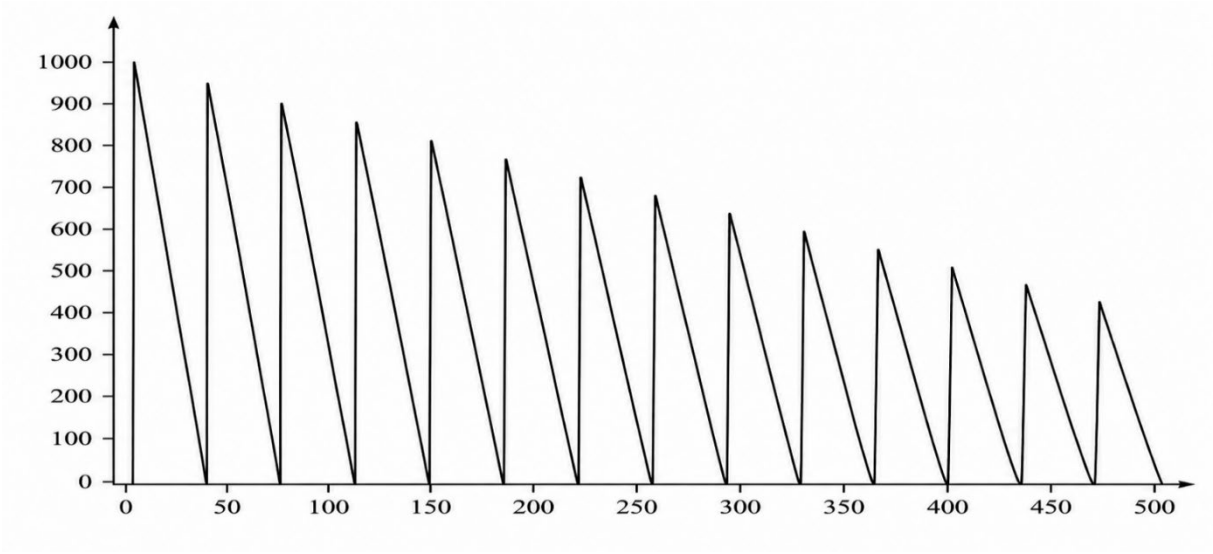


Fig.3: Dynamics of the inventory level q under an increased number of replenishment cycles $(n > n_0)$.

In this case, more frequent replenishments lead to lower order quantities and shorter replenishment intervals over the planning horizon T, resulting in smoother inventory dynamics.

The numerical implementation of the proposed algorithm was carried out using a custom-developed computer program written in Borland C++. The implementation includes the iterative linearization procedure based on Pshenichny's method, the computation of auxiliary quadratic programming subproblems, and the generation of graphical outputs illustrating inventory dynamics under different replenishment frequencies. The stopping criteria were defined by the tolerances $\varepsilon_1 = 10^{-2}$ and $\varepsilon_2 = 10^{-3}$, which were used to control the convergence of the iterative process.

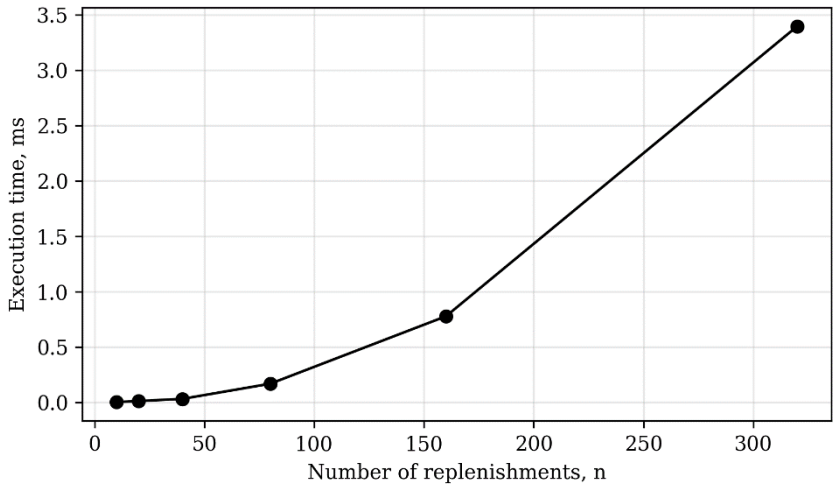


Fig.4: Execution time of the proposed algorithm for different values of n.

Figure 3 illustrates the dynamics of the inventory level qqq when the number of replenishment cycles is increased (n>n0). In this scenario, replenishments occur more frequently throughout the planning horizon T, resulting in smaller order quantities and shorter replenishment intervals.

As shown in the figure, the inventory level increases to a specified maximum level at each replenishment event and then decreases linearly as demand is satisfied over time. However, because replenishments are carried out more frequently, the duration of each cycle becomes shorter and the quantity ordered in each cycle is reduced. Consequently, inventory fluctuations occur with smaller amplitudes, leading to a more stable inventory profile.

Compared with the previous scenarios, variations in inventory levels are less pronounced. More frequent replenishments reduce the risk of stockouts and improve product availability, ensuring a more responsive supply system. On the other hand, the increased number of replenishment orders may lead to higher ordering and administrative costs due to the greater frequency of procurement activities.

The figure also indicates that the peak inventory levels gradually decrease over time, suggesting a progressive reduction in replenishment quantities or inventory requirements over the planning horizon. As a result, the inventory trajectory becomes smoother and reflects a more flexible inventory management policy.

Figure 3 demonstrates the impact of a higher replenishment frequency on inventory behavior. Smaller order quantities and shorter replenishment cycles contribute to smoother and more stable inventory dynamics, although these benefits may come at the expense of increased ordering and operational costs.

The results of the conducted research and numerical experiments demonstrate that the proposed linearization method for optimizing the modified Wilson EOQ model under a linear demand trend is highly effective both theoretically and practically. Within the framework of the study, the limitations of the classical Wilson model were eliminated, time-dependent linear demand behavior was incorporated, and a nonlinear optimization model reflecting more realistic economic conditions was developed.

The iterative algorithm based on Pshenichny's linearization method transformed the original nonlinear programming problem into a sequence of auxiliary quadratic programming subproblems. This approach significantly simplified the computational procedure, improved convergence stability, and reduced the computational difficulties typically associated with the direct solution of nonlinear systems.

The numerical results showed that the proposed method provides stable convergence for all considered cases and satisfies the stopping criterion within a limited number of iterations. This confirms both the finite-step solvability of the auxiliary quadratic programming subproblems and the high computational efficiency of the developed approach.

The computations further revealed that the minimum value of the total cost function is achieved in the vicinity of the theoretically predicted optimal number of orders $n = n_0$.

Deviations from this value resulted in an increase in the total logistics and inventory costs. In particular, when $n < n_0$, replenishment intervals became excessively long, leading to higher average inventory levels and increased holding costs. Conversely, when $n > n_0$, the replenishment frequency increased, generating additional ordering and transportation expenses. These findings clearly demonstrate the economic importance of accurately determining the optimal replenishment frequency.

The study also confirmed that the assumption of equal or approximately equal replenishment intervals under linearly increasing demand conditions provides sufficiently accurate approximations for practical applications. This justifies the use of simplified analytical EOQ solutions as effective initial approximations for nonlinear optimization procedures and contributes to faster computational performance.

The obtained results additionally indicate that the application of the dual quadratic programming framework enables the analytical computation of Lagrange multipliers, thereby improving the numerical stability of the optimization procedure. The incorporation of active constraints further ensures greater precision and robustness of the solution process.

From a practical perspective, the proposed model and algorithm are particularly relevant for food retail supply systems. In such systems, product perishability, demand variability, and accurate replenishment timing have a direct impact on the overall efficiency of the supply chain. The proposed approach enables the determination of optimal replenishment policies under dynamic demand conditions, more rational inventory management, and reduction of logistics-related costs.

Therefore, the results of the study demonstrate that:

- the modified Wilson EOQ model more adequately represents real economic processes under linear demand trends;
- the proposed linearization method ensures high computational stability in solving nonlinear optimization problems;
- auxiliary quadratic programming subproblems can be solved efficiently within a finite number of iterations;
- accurate determination of the optimal replenishment frequency plays a decisive role in minimizing total logistics costs;
- the assumption of equal replenishment intervals provides sufficiently accurate results for practical applications;
- the developed model and method possess significant practical applicability for inventory optimization in food retail supply chains.

The proposed approach can be regarded as an effective mathematical and computational tool for modern retail supply systems operating under dynamic demand conditions. Furthermore, the developed framework may be extended in future studies toward nonlinear demand functions, uncertainty modeling, and multi-product inventory systems.

5. Discussion

The obtained results demonstrate that the modified Wilson EOQ model with linearly changing demand provides an effective framework for inventory optimization under dynamic demand conditions. Unlike the classical EOQ model based on constant demand assumptions (Wilson, 1934), the proposed approach captures temporal variations in replenishment requirements and offers a more realistic representation of inventory processes in retail and food supply chain systems. Such demand dynamics are particularly important in environments characterized by fluctuating consumer preferences, seasonal effects, and short product life cycles (De Matteis and Mendoza, 1968; Silver and Meal, 1973; Sarkar, 2013).

The numerical analysis indicates that the minimum total inventory cost is achieved in the neighborhood of the optimal replenishment frequency. Even relatively small deviations from the optimal replenishment schedule lead to noticeable increases in holding, transportation, and ordering costs. Similar observations regarding the sensitivity of EOQ-type systems to replenishment policies have been reported in earlier inventory optimization studies (Battini et al., 2014; Dubelaar et al., 2001; Paul et al., 2018). In particular, insufficient replenishment frequency increases inventory holding periods and deterioration risk, whereas excessively frequent replenishment reduces operational efficiency due to additional transportation and setup costs (Boute et al., 2007; Sarkar, 2013).

Recent empirical studies further reinforce the importance of adaptive and resilient inventory policies under volatile operational environments. Luo et al. (2024) highlight that agile strategies significantly enhance responsiveness and cost efficiency in complex supply chain systems, while Kankoon and

Amornsawadwatana (2024) emphasize that robust inventory control mechanisms are critical for maintaining continuity during periods of global disruption. Similarly, Jama and Mohamud (2024) demonstrate that effective procurement practices contribute directly to improved organizational performance through better alignment between demand variability and resource allocation.

The results further show that equal or near-equal replenishment intervals provide sufficiently accurate approximations for practical applications under linearly increasing demand conditions. This observation is important from a managerial perspective, as it simplifies the implementation of the proposed approach in real-world retail systems involving multiple product categories and continuously changing inventory requirements. Similar approximation principles have previously been discussed in heuristic and time-varying demand models (Silver and Meal, 1973; Broyles et al., 2010). Consequently, the model can support inventory managers in balancing replenishment efficiency and operational simplicity (Fernie and Sparks, 2018).

An additional advantage of the proposed framework is its computational stability. The application of the Pshenichny linearization method transforms the original nonlinear optimization problem into a sequence of auxiliary quadratic programming subproblems, ensuring convergence of the iterative procedure under standard assumptions (Pshenichny, 1983; Pshenichny and Danilin, 1975). In contrast to approximation approaches based on truncated series expansions, the proposed method provides a mathematically justified optimization framework with stable numerical behavior. Such computational properties are particularly important for inventory systems requiring repeated recalibration under changing demand conditions (Isotupa, 2006; Minoux, 2018).

These findings are also consistent with broader studies emphasizing the importance of efficient inventory coordination and cost optimization in supply chain systems (Fleisch and Tellkamp, 2005; Tasdemir and Hiziroglu, 2019; Ortolani et al., 2011).

6. Conclusion

The main contribution of this paper is not limited to addressing limitations in Zeng et al. (2019), but rather lies in the development of a numerical optimization framework for solving nonlinear EOQ-type models. The proposed approach is designed for inventory systems characterized by time-dependent demand, nonlinear cost structures, and ordering constraints, and may provide a useful basis for further developments in inventory optimization and computational operations research.

Recent studies further indicate that the increasing complexity of modern supply chains requires more adaptive and resilient decision-support frameworks. For instance, Luo et al. (2024) emphasize that agile operational strategies significantly improve responsiveness in dynamic logistics environments, while Kankoon and Amornsawadwatana (2024) highlight the importance of robust inventory systems for ensuring continuity under global disruptions. Similarly, Jama and Mohamud (2024) demonstrate that procurement efficiency and organizational performance are strongly influenced by the ability to align decision-making processes with demand variability.

The modified EOQ models proposed by Zeng et al. (2019) extend the classical Wilson EOQ framework (Wilson, 1934) by considering linearly changing consumer demand over time. Such extensions are particularly relevant for retail supply chains and food distribution systems, where demand variability and inventory deterioration significantly affect replenishment decisions and total inventory costs (De Matteis and Mendoza, 1968; Silver and Meal, 1973; Sarkar, 2013).

The models developed by Zeng et al. (2019) are formulated as nonlinear optimization problems with linear constraints. Their solution methodology relies on truncated Maclaurin series expansions of nonlinear

terms in the objective function. Although this approximation simplifies the computational procedure, it does not provide theoretical guarantees of convergence to the exact optimal solution, especially in cases where nonlinear effects become more pronounced.

In contrast, the present study applies the linearization method developed by Pshenichny (1983) and Pshenichny and Danilin (1975), which provides a rigorous mathematical framework for solving the considered nonlinear optimization problems. The proposed approach transforms the original nonlinear model into a sequence of auxiliary quadratic programming subproblems and, under standard assumptions, ensures convergence to a stationary solution of the problem.

Furthermore, the study demonstrates the applicability of the proposed linearization approach to modified EOQ models with dynamic demand structures. The numerical example presented in this paper indicates that the method is capable of generating feasible and stable solutions under the considered parameter settings. These results suggest that the proposed framework shows potential for application to inventory optimization problems with nonlinear cost components and time-dependent demand. However, due to the limited number of numerical experiments, further computational studies and real-world case analyses are required to fully assess its robustness and practical applicability in modern retail supply systems.

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