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A queueing-inventory system with registration and orbital searching processes

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Abstract: In this paper, we consider a continuous time queueing-inventory system with customers' registrations and stochastic inventory on the replenishment policy of (s, S). Using the method of matrix analysis, we obtain the steady joint probability distribution of the number of customers in the orbit and the inventory level. Various system performance measures in the steady state are derived and the long-run total expected cost rate is calculated.

Keywords: Queueing-Inventory System, (s, S) Policy, Matrix Analytic Method, Registration, Retrial Customers

1. Introduction

Production-inventory systems have been extensively studied in the area of integrated supply chain management. Traditionally, this topic is usually investigated using queueing networks and multi-echelon inventory models. For details, one can see Berman (Berman, Kim, & Shimshak, 1993; Berman & Kim, 1999; Berman & Sapna, 2000) in which some integrated models appeared concerning the problem of how the classical performance measures are influenced by the management of attached inventory and vice versa: How inventory management has to react to queueing of demands and customers, which is due to incorporated service facilities. Recently, Schwarz et al. (Schwarz, et.al, 2006) studied an M/M/1 queueing system with inventory under continuous review and different inventory management policies, and with lost sales.

Artalejo (Artalejo, Krishnamoorthy, & Lopez-Herrero, 2006) studied inventory policies with positive lead-time and retrial of customers who could not get service during their earlier attempts to access the service station. Krishnamurthy (Krishnamoorthy & Jose, 2007) took a comparison of inventory systems with service, positive lead-time, loss, and retrial of customers. In their model, the retrial rate is assumed to be linear with the number of the customers in the orbit. By using the matrix analysis method (Latouche, & Ramaswami, 1999), they obtain a numeral value of the system. For further details, one can see Yang (Yang & Templeton, 1987) and Falin (Falin, 1990; Falin & Templeton, 1997).

In this paper, we will consider a continuous review queueing-inventory system with customers' registrations and the server is required to search for customers in the registration list according to first-come-first-serve (FCFS) discipline. The whole system combines the inventory with the server: the possible stochastic time in the system, such as the customer's inter-arrival time, the service time, the retrial time and the positive lead-time of the replenishment.

2. Problem Formulations and Analysis

We consider a continuous time retrial queueing system with stochastic inventory on the replenishment policy of (*s*, *S*). The customers arrive according to a Poisson process with the rate of α .The server is closed when the inventory is out of stock. As the (*s*, *S*) replenishment policy, when the on-hand inventory level drops to a prefixed level, say *s* (>0), an order for *Q* (=*S*-*s*>*s*) units is placed. The positive lead-time of the replenishment is exponential distribution with the rate β . The server is available to work when there are items in the inventory, the customer at the server will leave the system with one item from the inventory after a random service time, which we assume it is according to an exponential distribution with the rate.

If the arriving customer finds server unavailable, he will leave a message to register the system and then enter the retrial orbit. The server is required to search for these customers once they are available to provide service to them. It means the customers in the orbit will obtain the opportunity of being served after a random time, which is also assumed to be exponentially distributed with the rate θ . Compared to the customer in the orbit, new customer has priority to enter the server when they both attempt to enter the server at the same time; however, the priority is non-preemptive. The customers in the orbit take the retrial mechanism of FCFS, which means he who is the first coming in to the orbit, who is the first coming out to take the retrial chance. If he encounter the server is available to work, he will be served immediately, and otherwise he will return to the orbit.

At any moment t, let $N(t)$ denotes the number of customers in the orbit, $S(t)$ denotes the state of server 0, 1, 2, respectively, corresponding to the server is closed, idle, or at working) and $I(t)$ denotes the on-hand inventory level. The possible values of the above variables are: $N(t)\in\{0,1,2,...\}$, $I(t)\in\{0,1,2,..., S\}$.We assume that the above three random variables are independent of each other; we can conclude that the stochastic process $\{N(t),S(t),I(t)\}_{t\geq 0}$ is a three-dimensional Markov process. Its state space is listed as follows:

$$\Omega=\{(i,0,0):i\in N\}\cup\{(i,1,m):i\in N,m=1,2,...,S\}\cup\{(i,2,m):i\in N,m=1,2,...,S\}$$

The infinitesimal generator is $A=\left(a\left((i,j,k),(i^{'},j^{'},k^{'})\right)\right),(i,j,k)\in\Omega,(i^{'},j^{'},k^{'})\in\Omega$

By ordering the sets of state space as lexicographically, the infinitesimal generator A can be expressed in a block partitioned matrix with entries:

$$A=\begin{matrix}0\\1\\2\\\vdots\end{matrix}\begin{bmatrix}A_0 & C & & & \\ D & B & C & & \\ & D & B & C & \\ & & \ddots & \ddots & \ddots\end{bmatrix} \qquad (1)$$

It readily seen that $\{N(t),S(t),I(t)\}_{t\geq 0}$ is a level-independent QBD process.

3. Steady State Analysis

Under the stability condition, we can consider the steady probability of the system. Let $\pi_{i,j,k}=\lim_{t\to\infty}\Pr\{N(t)=i,S(t)=j,I(t)=k\}$ and let

$$\Pi=(\Pi_0,\Pi_1,\Pi_2,...),\Pi_i=(\pi_{i,1},\pi_{i,2})$$

$$\pi_{i,1}=(\pi_{i,0,0},\pi_{i,1,1},\pi_{i,1,2},...,\pi_{i,1,S}),\pi_{i,2}=(\pi_{i,2,1},\pi_{i,2,2},...,\pi_{i,2,S}),(i=0,1,2,...)$$

The vector Π satisfies: $\begin{cases}\Pi A=0\\ \Pi e=1\end{cases}$ (2)

From the well-known result on matrix-geometric methods (Latouche & Ramaswami, 1999); the steady-state probability vector Π can be given by:

$$\Pi_i=\Pi_0 R^i,(i=1,2,...), \qquad (3)$$

where the matrix R satisfies the matrix quadratic equation:

$$R^2D+RB+C=0, \qquad (4)$$

and the vector Π_0 can be obtained by solving the following equations

$$\begin{cases}\Pi_0(A_0+RD)=0\\ \Pi_0(I-R)^{-1}e=1\end{cases} \qquad (5)$$

From the matrix quadratic equation (4), due to the special structure of matrix C, the matrix R has the follow structure:

$$R=\begin{bmatrix}0_{(S+1)\times(S+1)} & 0_{(S+1)\times S}\\ R_1 & R_2\end{bmatrix},R_1=\left(r_{i,j}\right)_{S\times(S+1)},R_2=\left(t_{i,j}\right)_{S\times S}$$

Then, we can get

$$R^i = \begin{bmatrix} 0_{(S+1)\times(S+1)} & 0_{(S+1)\times S} \\ R_2^{\,i-1}R_1 & R_2^{\,i} \end{bmatrix}, (i = 0, 1, 2, ...)$$

Substituted it into the (4), the matrix quadratic equation can be re-written as:

$$\begin{bmatrix} 0_{(S+1)\times(S+1)} & 0_{(S+1)\times S} \\ R_1B_{11} + R_2B_{21} & R_2R_1D_1 + R_1B_{12} + R_2B_{22} + \alpha I_{S\times S} \end{bmatrix} = 0_{(2S+1)\times(2S+1)} \quad (6)$$

We can get a series of non-linear equations. Then, for any actual model, once the parameters are given, we can use some numeral iterative method, such as Gauss-Seidel method, to solve the numeral equations relevant to (5). All the probability vectors Π_i can be described as follow:

$$\Pi_i = \Pi_0 R^i = (\pi_{0,1}, \pi_{0,2}) \begin{bmatrix} 0_{(S+1)\times(S+1)} & 0_{(S+1)\times S} \\ R_2^{\,i-1}R_1 & R_2^{\,i} \end{bmatrix} = (\pi_{0,2}R_2^{\,i-1}R_1, \pi_{0,2}R_2^{\,i}), (i = 1, 2, ...)$$

We can partition the vector Π_i as

$$\Pi_i = (\pi_{i,1}, \pi_{i,2}), \text{ where } \begin{cases} \pi_{i,1} = \pi_{0,2}R_2^{\,i-1}R_1 \\ \pi_{i,2} = \pi_{0,2}R_2^{\,i} \end{cases}, (i \ge 1)$$

We can also use an alternative iterative method to solve the rate matrix R.

Let

$$\begin{cases} U = B + C\cdot G \\ G = (-U)^{-1}\cdot D \\ R = C\cdot(-U)^{-1} \\ U = B + R\cdot D \end{cases}, \text{ (Latouche \& Ramaswami, 1999)}$$

We can use them iteratively to obtain the rate matrix R.

Algorithm 1

$$G := (-B - C)\cdot D;$$

$$\textit{repeat}$$

$$\begin{cases} G_{old} := G; \\ U := B + C\cdot G; \\ G := G = (-U)^{-1}\cdot D \end{cases}$$

$$\textit{until} \ \ \|G - G_{old}\| \le \varepsilon$$

$$R := C\cdot(-U)^{-1}$$

Then, combined with (5), we can get a special non-zero solution vector Π_{0*} by solving the first matrix equation: $\Pi_0(A_0 + RD) = 0$; furthermore, together with the second equation: $\Pi_0(I - R)^{-1}e = 1$, we can get the boundary probability vector Π_0:

$$\Pi_0 = \left[\Pi_{0*}\cdot(I - R)^{-1}\cdot e\right]^{-1}\cdot\Pi_{0*}$$

Having the R and Π_0, we can obtain the rest probability vectors by the formula:

$$\Pi_i = \Pi_0 R^i, (i = 1, 2, ...)$$

4. System Performance Measures

(1) Expected Inventory Level: $\zeta_I = \sum_{i=0}^{\infty}\sum_{k=1}^{S} k(\pi_{i,1,k} + \pi_{i,2,k})$

(2) Expected Reorder Rate: $\zeta_R = \mu\sum_{i=0}^{\infty}\pi_{i,2,s+1}$

(3) Expected Number of Customers in the Orbit: $\zeta_O = \sum_{i=1}^{\infty} i\Pi_i e = \Pi_0 \sum_{i=1}^{\infty} iR^i e$

(4) Overall Rate of Losing Customers: $\zeta_L = \alpha P_{OFF} = \alpha\sum_{i=0}^{\infty}\pi_{i,0,0}$

(5) Overall Rate of Retrial: $\zeta_{OR} = \theta\cdot\sum_{i=1}^{\infty}\sum_{k=1}^{S}\left(\pi_{i,1,k} + \pi_{i,2,k}\right) = \theta\cdot\sum_{i=1}^{\infty}\left(\Pi_i\cdot e - \pi_{i,0,0}\right)$

(6) Successful Rate of Retrial: $\zeta_{ORS} = \theta\sum_{i=1}^{\infty}\sum_{k=1}^{S}\pi_{i,1,k} = \theta\left[\pi_{0,2}(I-R_2)^{-1}R_1 e - P_{OFF}\right]$

(7) Fraction of Successful Rate of Retrial: $\zeta_{FORS} = \frac{\zeta_{ORS}}{\zeta_{OR}}$

5. Cost Analysis

Let $TC(s,S)$ denote the long-run expected cost rate under the following cost structure.

c_O : Waiting cost of a customer in the orbit per unit time；

c_I : The inventory carrying cost per unit item per unit time；

c_R : Setup cost per order.

Then we have $TC(s,S) = c_O\zeta_O + c_I\zeta_I + c_R\zeta_R$

Based on the equations obtained in Section 4, we get that:

$$TC(s,S) = c_O\Pi_0 R(I-R)^{-2}e + c_I\sum_{i=0}^{\infty}\sum_{k=1}^{S}k(\pi_{i,1,k} + \pi_{i,2,k}) + c_R\mu\sum_{i=0}^{\infty}\pi_{i,2,s+1}$$

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