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A linear programming approximation for network capacity control problem with customer choice

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Abstract: By dividing customers into different segments according to origin-destination (O-D) pairs, we consider a network capacity control problem where each customer chooses the open product within the segment he belongs to. Starting with a Markov decision process (MDP) formulation, we approximate the value function with an affine function of the state vector and develop our model based on the O-D demands. The affine function approximation results in a linear program (LP). We give a column generation procedure for solving the dual problem of the LP and provide the numerical results.

Keywords: network revenue management, choice behavior, dynamic programming, linear programming

1. Introduction

In most of capacity control models of network revenue management, uncertain demands are considered for each product (each product for a specific fare class). However, the exploration of models based on stochastic demands between O-D pairs will probably become increasingly important as opportunities for code-sharing within strategic partnerships increases the breadth of choice in customers itinerary selections. Motivated by this consideration, Liu et al. (2011) provided an independent demand model which is developed with O-D demands. This paper will focus on extending the model of Liu et al. (2011) to the customer choice setting.

The remainder of the paper is organized as follows. Section 2 provides a brief overview of the related literature. Section 3 gives a Markov decision process formulation. Section 4 considers the affine functional approximation and the resulting problem, derives the LP model from it. Section 5 gives a column generation algorithm. We provide numerical results in Section 6. Summary is presented in the final section.

2. Brief Review of Literature

Belobaba and Hopperstad (1999) demonstrate the significant impact of passenger choice behavior on the performance of revenue management systems. Van Ryzin and Vulcano (2008) propose a simulation-based optimization approach to network capacity control problem under a general choice scheme. Gallego et al. (2004) provide a choice-based deterministic linear programming (CDLP) model. Motivated by the work of Gallego et al. (2004), Liu and van Ryzin (2008) study a linear programming formulation and provide a column generation algorithm to solve the problem for the multinomial logit (MNL) choice model with disjoint consideration segments. Bront et al. (2009) focus on the MNL choice model with overlapping consideration sets. They also provide a column generation algorithm to solve it.

3. Problem Formulation

We begin with a flight network, which is comprised of m flight legs, indexed by the set $i \in I = \{1, \ldots, m\}$. The network has l O-D pairs. The set of O-D pairs in the entire network is denoted by $n \in N = \{1, \ldots, l\}$. Flight legs can be combined to create routes which serve various O-D pairs in the network. Typically, there are multiple routes that can serve a given O-D pair. The firm sells k products. The set of products is denoted by $j \in J = \{1, \ldots, k\}$. Let $J_n \subseteq J$ be the set of products which belong to O-D pair n, then $J = \bigcup_{n\in N} J_n$. Furthermore, we have $J_n \cap J_{n'} = \varnothing$ for $n \neq n'$. The fare for product j is f_j.

Define the incidence matrix $A = [a_{i,j}]$, where $a_{i,j} = 1$ if product j uses leg i and $a_{i,j} = 0$ otherwise; the j th column of A, denoted A^j, is the incidence vector for product j. We let A^j denote the set of legs used by product j.

Time is discrete, there are T periods, and the index t represents an arbitrary time. Within each time period t, at most one customer arrives. The probability of having an arrival in each time period is denoted by λ.

Let $S \subseteq J$ be the set of the total available products which are offered by the firm. Given the set S, let $P_{nj}(S)$ denote the probability that a segment-n customer chooses the product $j \in J_n \cap S$. To determine the purchase probability $P_{nj}(S)$, define a preference vector $v_n \geq 0$, which indicates the customer "preference weight" for each product contained in J_n, and the no-purchase preference value v_{n0}. Then $P_{nj}(S) = v_{nj} \Big/ \left(\sum_{j\in J_n \cap S} v_{nj} + v_{n0} \right)$.

If $j \notin J_n \cap S$ or $j \notin J_n$, then $v_{nj} = 0$. Let $P_j(S)$ be the probability that the product $j \in S$ is chosen by an arriving customer. Noting that the seller ex ante cannot distinguish which segment each arriving customer belongs to, then $P_j(S) = \sum_{n\in N} p_n P_{nj}(S)$.

The state of the network is described by a vector $x = (x_1, ..., x_m)$ of remaining leg capacities; the initial state is denoted by vector $c = (c_1, ..., c_m)$.If a single unit of product $j \in S$ is sold, the state of the network changes to $x - A^j$.

Let $v_t(x)$ be the maximum total expected revenue over periods $t,\dots,T$ starting at state x at the beginning of period t. Then $v_t(x)$ must satisfy the Bellman equations with the boundary condition $v_{T+1}(x)=0\ \forall x$.

$$v_t(x)=\max_{S\subseteq J(x)}\left\{\sum_{n\in N}\lambda_n\sum_{j\in S}P_{nj}(S)\left[f_j-\left(v_{t+1}(x)-v_{t+1}\left(x-A^j\right)\right)\right]\right\}+v_{t+1}(x),\quad \forall t,x\in X_t \tag{1}$$

The set $J(x)=\{j\in J: x\geq A^j\}$ is the set of products that can be offered when the state is x.

The value function at initial state c can be computed by the linear program

(P0) $\min_{v(\bullet)}\ v_1(c)$

$$\text{s.t.}\quad v_t(x)\geq\sum_{n\in N}\lambda_n\sum_{j\in S}P_{nj}(S)\left[f_j-\left(v_{t+1}(x)-v_{t+1}\left(x-A^j\right)\right)\right]+v_{t+1}(x),\quad \forall t,x\in X_t,S\subseteq J(x)$$

with decision variables $v_t(x)\ \forall t,x$.

4. Functional Approximation

In general, (1) and (P0) are intractable because of the enormous size of the state space. In this section, we first use a set of affine functions to approximate $v_t(\cdot)$, and then give the resulting primal-dual formulations.

Consider the affine functional approximation

$$v_t(x)\approx\theta_t+\sum_{i\in I}\pi_{t,i}x_i, \tag{2}$$

where θ_t is a constant offset and $\pi_{t,i}$ estimates the marginal value of a seat on leg i in period t. We assume $\theta_{T+1}=0$ and $\pi_{T+1,i}=0,\ \forall i$.

Plugging (2) into (P0) yields that

(P1) $\min_{\theta,\pi}\theta_1+\sum_{i\in I}\pi_{1,i}c_i$

$$\text{s.t. } \theta_t - \theta_{t+1} + \sum_{i\in I}\left[\pi_{t,i}x_i - \pi_{t+1,i}\left(x_i - \sum_{n\in N}\lambda_n \sum_{j\in S} P_{nj}(S)a_{i,j}\right)\right] \ge \sum_{n\in N}\lambda_n \sum_{j\in S} P_{nj}(S) f_j,\ \ \forall t, x\in X_t, S\subseteq J(x).$$

The dual of (P1) is

$$\text{(D1)}\ \ Z_{\mathrm{D1}} = \max_{\gamma} \sum_{t,x\in X_t, S\subseteq J(x)} \left(\sum_{n\in N}\lambda_n \sum_{j\in S} P_{nj}(S) f_j\right)\gamma_{t,x,S} \qquad (3)$$

$$\text{s.t. } \sum_{x\in X_t, S\subseteq J(x)} x_i \gamma_{t,x,S} = \begin{cases} c_i & \text{if } t=1, \\ \displaystyle\sum_{x\in X_{t-1}, S\subseteq J(x)} \left(x_i - \sum_{n\in N}\lambda_n \sum_{j\in S} P_{nj}(S)a_{i,j}\right)\gamma_{t-1,x,S} & \forall t = 2,\dots,T, \end{cases} \quad \forall i,t, \qquad (4)$$

$$\sum_{x\in X_t, S\subseteq J(x)} \gamma_{t,x,S} = \begin{cases} 1 & \text{if } t=1, \\ \displaystyle\sum_{x\in X_{t-1}, S\subseteq J(x)} \gamma_{t-1,x,S}, & \forall t = 2,\dots,T, \end{cases} \qquad (5)$$

$$\gamma \ge 0.$$

5. Column Generation Algorithm

The program (D1) has a large number of variables but relatively few constraints, so we can solve it via column generation. Denote the reduced profit of $\gamma_{t,x,S}$ by

$$\omega_{t,x,S} = \sum_{n\in N}\lambda_n \sum_{j\in S} P_{nj}(S) f_j - \sum_{i\in I}\left[\pi_{t,i}x_i - \pi_{t+1,i}\left(x_i - \sum_{n\in N}\lambda_n \sum_{j\in S} P_{nj}(S)a_{i,j}\right)\right] - \theta_t + \theta_{t+1}.$$

Given a feasible solution to (D1), denoting the resulting prices by θ, π, now solve

$$\max_{t,x\in X_t, S\subseteq J(x)} \omega_{t,x,S} = \max_{t,x\in X_t, S\subseteq J(x)} \sum_{n\in N}\lambda_n \sum_{j\in S} P_{nj}(S)\left(f_j - \sum_{i\in I} a_{i,j}\pi_{t+1,i}\right) - \sum_{i\in I}\left(\pi_{t,i} - \pi_{t+1,i}\right)x_i - \theta_t + \theta_{t+1}.$$

Let the binary vector $u \in \{0,1\}^k$ be the characteristic vector of set S. It indicates which products are offered at any period, $u_j = 1$ if $j\in S$, and $u_j = 0$ otherwise. For fixed $t > 1$, this is equivalent to solving the following optimization problem:

$$\text{(S1)}\qquad \max_{x,u} \sum_{n\in N}\lambda_n \frac{\sum_{j\in J_n} u_j v_{nj}\left(f_j - \sum_{i\in I} a_{i,j}\pi_{t+1,i}\right)}{\sum_{j\in J_n} u_j v_{nj} + v_{n0}} - \sum_{i\in I}\left(\pi_{t,i} - \pi_{t+1,i}\right)x_i - \theta_t + \theta_{t+1}$$

$$\begin{aligned} \text{s.t. } & a_{i,j}u_j \le x_i, && \forall i,j, \\ & u_j \in \{0,1\}, && \forall j, \\ & x_i \in \{0,\dots,c_i\}, && \forall i. \end{aligned}$$

(S1) can be transformed into a mixed integer linear programming problem (see Bront et al. 2009) and solved by any mixed integer programming software package. If the optimal function value is nonpositive, then we have attained optimality; otherwise, we add the column to the existing set of columns for (D1).

Let (π^*, θ^*) be the optimal solution for (P1). A control policy in period t and state x can be computed by solving

$$\max_{u_j \in \{0,1\{x \geq A^j\}\} \forall j \in J} \sum_{n \in N} \lambda_n \frac{\sum_{j \in J_n} u_j v_{nj} \left(f_j - \sum_{i \in I} a_{i,j} \pi^*_{t+1,i} \right)}{\sum_{j \in J_n} u_j v_{nj} + v_{n0}}. \qquad (6)$$

We can solve (6) using simple ranking procedure (see Liu and Van Ryzin 2008).

6. Numerical Results

Figure 1 illustrates a hypothetical airline network which consists of five legs, six O-D pairs and ten routes. Furthermore, two fare classes (Business and Leisure) are offered for each route. Business fares are drawn from the Poisson distribution with mean 200 and Leisure fares are drawn from the Poisson distribution with mean 100. For simplicity, we considered stationary demands with the probability 0.2 for having no customer arrival in a period. We generated problem instances with $T \in \{20, 50, 100, 200, 500\}$. For each instance, we set the initial capacity, c, to be the same for each leg. We solved (D1) and simulated each instance 100 times for each policy, using the same sequence of customer demands across different policies. The results are shown in Table1.

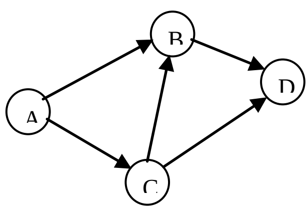

Fig.1: Hypothetical airline network with five legs, six O-D pairs and ten routes.

Table 1: Policy results

Capacity	Mean	Std.err.

T	per leg		
20	3	1456.40	135.21
50	7	3468.40	302.75
100	15	9132.80	608.42
200	29	13480.00	985.34
500	76	33208.00	1362.20

7. Summary

In this paper, we consider a network capacity control problem where customers choose the open product according to their O-D pair. Starting with a Markov decision process (MDP) formulation, we make an affine functional approximation to the optimal dynamic programming value function. Then, we derive the program (D1). We give a column generation procedure for solving (D1) and provide the numerical results.

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