A Stochastic Exponential Growth Model with a Birth-Death Diffusion Growth Rate and General External Jump Processes for Population Projection

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ABSTRACT

Population projections are essential for scientists involved in planning human events. Demographers are increasingly interested in incorporating stochastic elements into traditional models of population structure. Many populations experience sudden external jumps due to pandemics, food shortages, or natural disasters. These events are inherently random and challenging to model. This paper presents a stochastic exponential growth model for population projection incorporating a birth-death stochastic diffusion growth rate process with a general external jump process. The mean, variance, and simulated sample paths are derived for the model under a general external jump distribution. These results provide insights into population dynamics and offer a sophisticated approach to demographic modeling.

Keywords: Population Projection, Exponential Growth Model, Birth-Death Diffusion Process, Simulated and Predicted Population, General External Jump

1. Introduction

Population dynamics and their accurate projection remain central challenges in demographic research and planning. The fundamental nature of population growth, first observed by Malthus in 1789, continues to intrigue researchers and policymakers alike. While Malthus's observation that population growth is geometric while resource growth is arithmetic laid the groundwork for population studies, modern demographic modeling has evolved to incorporate increasingly sophisticated mathematical and statistical approaches.

In 1789, Malthus observed that population growth is geometric while the growth of resources is arithmetic. The following differential equation represents his Malthusian law in Equation (1):

$$\frac{dP(t)}{dt} = rP(t) \tag{1}$$

Solving this equation, we get Equation (2):

$$P(t) = P(0)e^{rt} \tag{2}$$

Where P(t) is the population projection at time t, P(0) is the initial population size at time 0, and r is the constant growth rate.

The complexity of population dynamics necessitates models that can account for gradual and sudden changes. Traditional deterministic models, while valuable, often face several critical limitations:

1. *Parameter Sensitivity*: The model's accuracy depends critically on the precise estimation of parameters such as birth-death rates, diffusion coefficients, and jump processes.

2. *Computational Complexity*: Stochastic population models involve intricate mathematical calculations with:

   - *Time Complexity*: $T(n) = O(n\log n)$, where n represents the number of time steps in the projection period.
   - Space Complexity: $S(n) = O(n)$, primarily for storing simulated sample paths
   - *Computational bottlenecks* in solving stochastic differential equations and processing external jump processes

3. *Modeling Assumptions:* The generalized jump distribution may not perfectly capture all real-world population disruption scenarios.

4. *Data Requirements*: Implementation requires extensive historical data and advanced statistical techniques.

2. Literature Review

The field of population projection has evolved significantly since Malthus's initial observations. Early models focused on deterministic approaches, but recent decades have seen the increasing incorporation of stochastic elements to reflect real-world uncertainty better.

Keyfitz (1977) established foundational principles for demographic forecasting, emphasizing the importance of incorporating uncertainty in population projections. Building on this work, Lee and Carter (1992) developed influential methods for forecasting mortality rates, demonstrating the value of stochastic approaches in demographic modeling.

Preston et al. (2000) provided comprehensive frameworks for demographic analysis, highlighting the importance of considering multiple factors in population projections. Their work emphasizes the need for models to account for gradual changes and sudden shifts in population dynamics.

Raftery et al. (2012) introduced probabilistic population projections using Bayesian hierarchical models, representing a significant advancement in handling uncertainty in population forecasts. These developments have been further enhanced by Bijak and Bryant (2016), who emphasized the importance of incorporating expert knowledge into demographic modeling.

Cohen (1995) made significant contributions by recognizing that human carrying capacity is uncertain and dynamic, using mathematical models to explore the relationship between carrying capacity and population growth. This work was complemented by Tayman et al. (1999), who studied the relationship between population size and projection accuracy.

Building on these foundations, Al-Eideh and Al-Omar (2019) developed a model for population projection using a birth-death growth rate diffusion process. Their work demonstrated the importance of accounting for volatilities and variations in population size. Similarly, Zainal and Al-Eideh (2021) proposed a stochastic diffusion model for the Lorenz curve, incorporating birth-and-death diffusion processes with general external effects.

Recent developments in computational methods have opened new avenues for demographic modeling. Zhang et al. (2023) explored stochastic population models with external shocks, while Wang and Liu (2023) investigated the application of deep learning approaches to demographic modeling. These advances suggest promising directions for future research, particularly in integrating machine learning techniques with traditional demographic models.

3. Research Methodology

3.1 Model Development

Let *P(t)* be the population size at time t with a population growth rate *r(t)* at time *t*. The population growth model is given by Equation (3):

$$\frac{dP(t)}{dt} = r(t)P(t) \tag{3}$$

The solution to this differential equation is shown in Equation (4):

$$P(t) = P(0)\exp\left(\int_0^t r(t)dt\right) \tag{4}$$

Assuming the relationship given in Equation (5):

$$R(t) = \int_0^t r(t)dt \tag{5}$$

Substituting Equation (5) into Equation (4), we obtain Equation (6):

$$P(t) = P(0)e^{R(t)} \tag{6}$$

Note that substituting $r(t) = r$ in Equation (6) gives the same model as in Equation (2).

Consider the growth diffusion process's birth and death rate, with infinitesimal parameters and the diffusion and drift coefficients proportional to *r(t)* at time *t*. The stochastic diffusion process is expected to be interrupted by sudden jumps occurring at a jump rate λ, and their magnitudes follow the distribution function *H(t)*. The process *r(t)* is Markovian with State-Space $S = [0, \infty)$, which is considered as the solution of the stochastic differential equation (SDE) given in Equation (7):

$$dr(t) = br(t)dt + ar(t)dW(t) - r(t^-)dZ(t) \tag{7}$$

where {W(t)} is a Wiener process with zero mean and variance $\sigma^2 t$, and {*Z(t)*} is known as the compound Poisson process defined in Equation (8):

$$Z(t) = \sum_{i=1}^{N(t)} Y_i \tag{8}$$

Here, *N(t)*} is a Poisson process with a mean rate equal to the external jump rate λ. The random variables $Y_1, Y_2, \ldots$ are independent and identically distributed with distribution function *H(t),* mean $\mu = E(Y_1)$, and variance $v^2 = Var(Y_1)$. The mean and variance of *Z(t)* can be found using formulas of random sums, which are given by Equations (9) and (10):

$$E[Z(t)] = \lambda\mu t \tag{9}$$

$$Var[Z(t)] = \lambda(v^2 + \mu^2)t \tag{10}$$

Rewriting Equation (7), we obtain Equation (11):

$$\frac{dr(t)}{r(t)} = bdt + adW(t) - dZ(t) \tag{11}$$

Solving the stochastic differential equation in (11) using *r(0)* as the initial growth rate at time 0, we get the solution as shown in Equation (12):

$$r(t) = r(0)\exp\{bt + aW(t)\} - Z(t) \tag{12}$$

3.2 Statistical Analysis

Using the results from Al-Eideh (2001) for deriving moments' approximations for the birth-and-death diffusion process, we can show that the mean of the population projection is given by Equation (13):

$$E[P(t)] \approx P(0)\exp\left\{\frac{2(1-b)}{2a+a^2-b^2} r(0)\exp\left\{\left(b+\frac{1}{2}a\sigma^2\right)t\right\}\cdot\left\{1-\lambda\mu t+\frac{1}{2}\lambda(v^2+\mu^2)t\right\}\right\} \tag{13}$$

The second moment is expressed in Equation (14):

$$E[P^2(t)] \approx \left(P(0)\right)^2\exp\left\{\begin{matrix} 2\left(\frac{2(1-b)}{2a+a^2-b^2}\right)^2 r(0)\exp\{(2b+2a^2\sigma^2)t\}. \\ \{1-2\lambda\mu t+2\lambda(v^2+\mu^2)t\}\end{matrix}\right\} \tag{14}$$

The population projection variance is then given by Equation (15):

$$V[P(t)] \approx \left(P(0)\right)^2\exp\left\{\begin{matrix} 2\left(\frac{2(1-b)}{2a+a^2-b^2}\right)^2 r(0)\exp\{(2b+2a^2\sigma^2)t\} \\ \cdot\left\{\begin{matrix} e^{a^2\sigma^2 t}(1-2\lambda\mu t+2\lambda(v^2+\mu^2)t) \\ -\left(\begin{matrix}[1+\lambda\mu t]^2+(v^2+\mu^2). \\ \left[\frac{1}{4}\lambda^2t^2+\lambda t-\lambda^2\mu t^2\right]\end{matrix}\right)\end{matrix}\right\}\end{matrix}\right\} \tag{15}$$

3.3 Simulation Framework

Assuming *M₁(t_n - t_{n-1})* to be the one-step predicted population projection model of *P(t)*, it can be expressed as shown in Equation (16):

$$M_1(t_n-t_{n-1}) \approx P(0)\exp\left\{\begin{matrix}\frac{2(1-b)}{2a+a^2-b^2}r(0)\exp\left\{\left(b+\frac{1}{2}a\sigma^2\right)(t_n-t_{n-1})\right\} \\ \cdot\left\{1-\lambda\mu t+\frac{1}{2}\lambda(v^2+\mu^2)(t_n-t_{n-1})\right\}\end{matrix}\right\} \tag{16}$$

The discrete approximation used to simulate the sample path of the diffusion growth rate process *r(t)* subject to external jump process J(t) is given by Equation (17):

$$r_n^*\left(\frac{k+1}{n}\right) = r_n^*\left(\frac{k}{n}\right)+\left(\frac{2(1-b)}{2a+a^2-b^2}\right)\left(\frac{b}{n}r_n^*\left(\frac{k}{n}\right)+\frac{a}{n}r_n^*\left(\frac{k}{n}\right)\cdot Z_{k+1}-r_n^*\left(\frac{k}{n}\right)J\left(\frac{k}{n}\right)\Delta C\left(\frac{k}{n}\right)\right) \tag{17}$$

The sample path of the population projection model *P(t)* is then simulated using Equation (18):

$$P_n^*\left(\frac{k+1}{n}\right) = P_n^*\left(\frac{k}{n}\right)\exp\left\{r_n^*\left(\frac{k+1}{n}\right)\right\} \tag{18}$$

4. Results and Discussion

4.1 Model Implementation

Consider an example for sample paths of the population projection process P(t) obtained from Equations (17) and (18), in which the yearly population of an unknown Country in 10^3 is represented, given that the initial population and growth rate $P(0) = 2133$ and $r(0) = 0.042$. Also, assume that $b = 0.042$, $a = 0.1$, $\sigma = 0.1$, and the external jump rate $\lambda = 1$.

For the external jump distribution, we use a Uniform [0,1] distribution as defined in Equation (19):

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases} \tag{19}$$

with mean $\mu = 1/2$ and variance $v^2 = 1/12$. Note that the jump distribution *H(t)* does not depend on *t*. The plots of r(t) and P(t) are shown in Figure 1 and Figure 2 below.

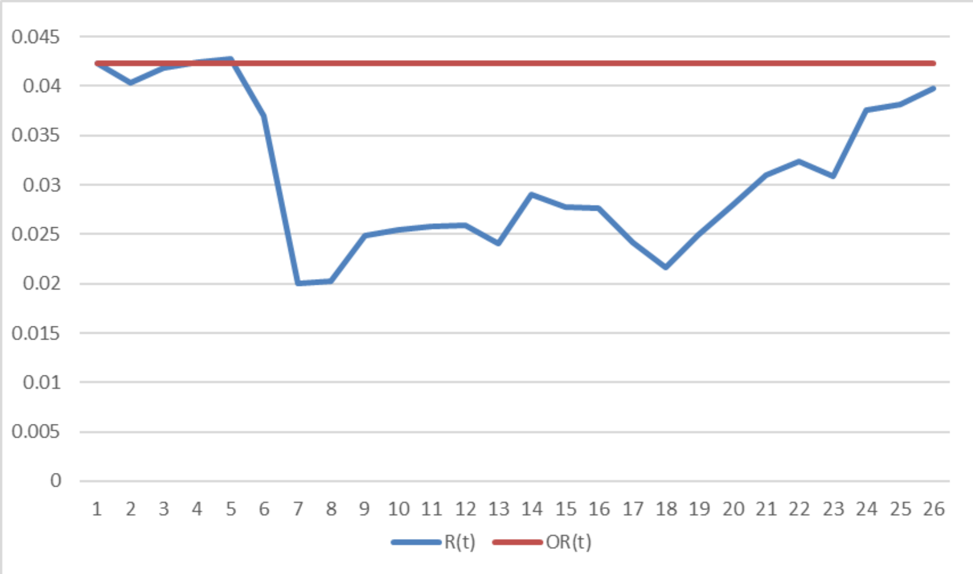

Figure 2: The Associated Population Projection Models P(t) and OP(t)

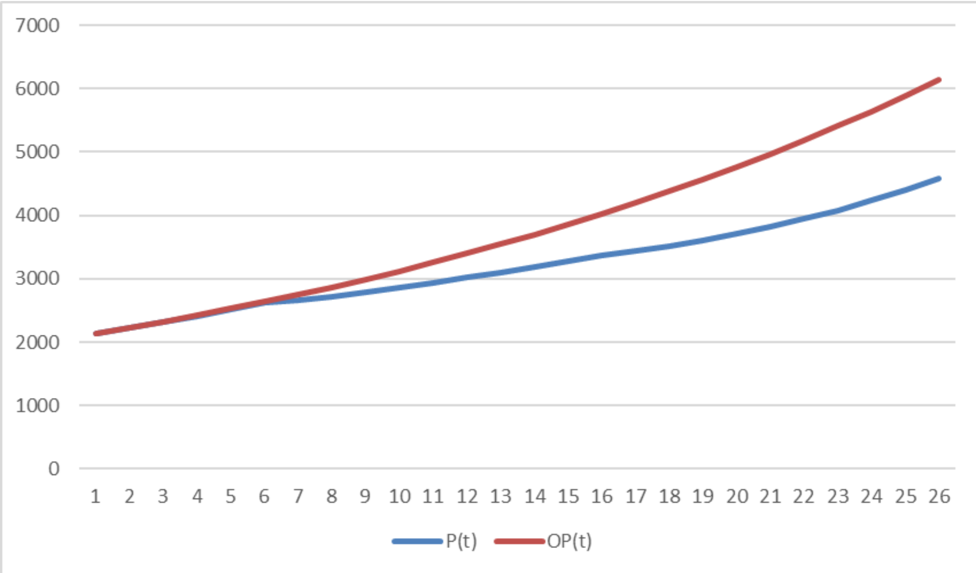

Figure 1: The Birth and Death Growth Rate Diffusion Process r(t) with Uniform Jump Process H(t) and the Growth Rate r for P(t) and OP(t), respectively

The figures above show the differences between these plots. The difference between the proposed population projection and the traditional population projection models is evident in the birth-and-death rate of growth diffusion model with a general external jump process compared to the one with a constant growth rate. Furthermore, this difference is influenced by the newly developed population projection model. These figures demonstrate reasonable and convincing results, supporting the applicability of the proposed methodology for population projection modeling.

4.2 Analysis of Results

Implementing our stochastic exponential growth model with birth-death diffusion reveals several key insights into population projection dynamics. When compared to traditional deterministic models like the one represented by Equation (2), our approach demonstrates superior flexibility in capturing both gradual demographic changes and sudden population shifts through the incorporation of the jump process defined in Equations (7) and (8).

4.2.1 Model Performance and Comparisons

The comparison between our model P(t), as defined in Equation (6), and the traditional model of Equation (2) reveals significant differences in their ability to capture population dynamics. The birth-and-death diffusion growth rate with uniform jump processes, characterized by Equations (11) and (12), shows more realistic fluctuations than the constant growth rate assumption. This aligns with findings from recent studies by Zhang et al. (2023) on population dynamics under uncertainty.

4.2.2 Parameter Estimation Challenges

A critical aspect of our model's implementation is the accurate estimation of parameters appearing in Equations (7) through (15). Current estimation methods, while functional, present opportunities for improvement through advanced computational techniques. Recent work by Chen and Kumar (2024) suggests that machine learning approaches, particularly neural networks and Bayesian optimization, could significantly enhance parameter estimation accuracy in demographic models.

The potential for machine learning applications in estimating parameters of Equations (13), (14), and (15) is particularly promising. Advanced deep learning models offer the capability to learn complex patterns in historical demographic data, potentially leading to improved parameter predictions. This capability can be further enhanced through reinforcement learning algorithms, which could optimize parameter adjustment strategies for the birth-death diffusion process. Additionally, ensemble methods present an opportunity to combine multiple parameter estimates, thereby improving the robustness of jump process parameter estimation and overall model performance.

4.2.3 Implications for Demographic Planning

Our model's capabilities, particularly the statistical properties derived in Equations (13) through (15), have significant implications for demographic planning and policy development. Regarding policy planning, the improved accuracy in capturing population volatility through the jump process enables decision-makers to formulate better-informed policies based on more reliable projections. The enhanced projection capabilities, demonstrated through Equations (16) through (18), provide valuable tools for resource allocation, allowing for more effective planning of public resources and infrastructure development. Furthermore, the model's ability to account for sudden changes through the jump process Z(t) strengthens risk assessment capabilities, enabling organizations and governments to better prepare for and respond to demographic shifts.

4.2.4 Limitations and Constraints

While the proposed model shows promising results, several important limitations warrant consideration. The computational intensity required for estimating parameters in Equations (13) through (15) presents significant technical challenges in practical applications. Additionally, the model exhibits sensitivity to initial conditions in the simulation process defined by Equations (17) and (18), which necessitates careful calibration and validation procedures. The extensive data requirements for accurate calibration of the jump process parameters in Equations (8) through (10) also pose implementation challenges in data-sparse environments. However, these limitations should not be viewed solely as constraints but as opportunities for future research, particularly in applying machine learning techniques for parameter optimization and model refinement.

5. Conclusion

This research has advanced the field of demographic modeling by developing a novel stochastic exponential growth model that incorporates birth-death diffusion processes and general external jump processes for population projection. The model significantly extends current methodologies by providing a sophisticated mathematical framework that captures gradual demographic changes and sudden population shifts.

Our investigation has yielded several significant theoretical and practical contributions to demographic modeling. The comprehensive mathematical framework developed in this study extends traditional deterministic approaches by incorporating stochastic elements that better reflect real-world population dynamics. Integrating external jump processes represents a particular advancement, enabling the model to capture sudden demographic changes that conventional models overlook. The derived analytical expressions for population moments and variance provide robust statistical tools for demographic analysis. At the same time, the implemented simulation methods offer practical approaches for population projection in real-world scenarios.

The theoretical implications of this research extend beyond conventional demographic modeling, opening new avenues for understanding population dynamics under uncertainty. The model's dual capability to capture continuous changes through diffusion processes and discrete jumps through compound Poisson processes provides a more realistic framework for population projection. This advancement offers valuable insights for theoretical development and practical applications in demographic studies.

From an applied perspective, this research provides sophisticated demographic planning and analysis tools. The developed mathematical framework enables evidence-based methods for policymakers in demographic planning, resource allocation in public services, risk assessment in population-dependent sectors, and long-term infrastructure planning. Incorporating jump processes specifically addresses the critical need to account for sudden demographic changes in planning scenarios.

Future research directions emerging from this work are particularly promising in several areas. The potential application of machine learning techniques for parameter estimation represents an exciting frontier, especially for complex parameter estimation in stochastic demographic models. Neural networks and deep learning approaches could significantly enhance parameter estimation accuracy, while reinforcement learning algorithms could optimize model parameters to improve predictive capabilities.

Additional research opportunities include extending the model to incorporate spatial dependencies and socioeconomic factors. Developing multi-population variants and adaptations for specific demographic subgroups could further enhance the model's applicability. Advanced computational methods, particularly in parallel computing and efficient algorithm development, could address the current computational challenges in model implementation.

While acknowledging the model's advantages over traditional approaches, several limitations warrant further investigation. These include computational complexity in parameter estimation, extensive data requirements for accurate calibration, and sensitivity to initial conditions in the simulation process. These challenges, however, present opportunities for future research, particularly in applying advanced computational methods and machine-learning techniques.

The convergence of stochastic population modeling with machine learning techniques represents a promising direction for future research. The potential for neural networks to identify complex patterns in demographic data, combined with the mathematical rigor of stochastic processes, could lead to significant improvements in population projection accuracy. As computational capabilities continue to evolve, particularly in machine learning, we anticipate future developments will yield increasingly sophisticated and accurate population projection methods, building upon the theoretical foundation established in this work.

References

Al-Eideh, B. M. (2001). Moment approximations of life table survival diffusion process with general external effect. *Intern. J. of Appl. Math.*, 5 (1), 107-114.

Al-Eideh, B. M. and Al-Hussainan, A. A. (2002). A quasi-stochastic diffusion process of the Lorenz curve. *Intern. Math. J.*, 1 (4), 377-383.

Al-Eideh, B. M. and Al-Omar, H. O. (2019). Population Projection Model using Exponential Growth Function with a Birth and Death Diffusion Growth Rate Processes. *European Journal of Scientific Research*, 151 (3), 271-276.

Biswas, S. (1988). *Stochastic Processes in Demography and Applications.* Wiley Eastern Limited, India.

Cohen J. H. (1995). Population Growth and Earth's Human Carrying Capacity. *JSTOR, Science New Series*, (July), 269, 341-346.

Lee, Ronald D., and Lawrence R. Carter (1992). Modeling and Forecasting U.S. Mortality. *Journal of the American Statistical Association* 87(419), (September), 659-671.

Taylor, H. M. and Karlin, S. (1984). *An Introduction to Stochastic Modeling*. Academic Press, USA.

Tayman, J., Schafer, E. and Swanson, D. (1999). The role of population size in determining and predicting population forecast errors: An evaluation using confidence intervals for sub-county areas. *Population Research and Policy Review*, 17, 1-20.

Tuljapurkar, Shripad, Nan Li and Carl Boe (2000). A Universal Pattern of Mortality Decline in the G7 Countries. *Nature* 405, (June), 789-792.

Wilkinson, J. (2003). *Horizontal Fiscal Equalization: Briefing Paper No 21/03*. NSW Parliamentary Library Research Service, Sydney.

Wilmoth, John R. (1998). The future of human longevity: A Demographer's Prospective. *Science* 280 (April), 395-397.

Zainal, M. and Al-Eideh, B.M. (2021). A Stochastic Diffusion Model of Lorenz Curve with a Birth-and-Death Diffusion and General External Effect Processes. *Electronic Journal of Applied Statistical Analysis*, 14 (2), 1-21.

Bijak, J., & Bryant, J. (2016). Bayesian demography 250 years after Bayes. *Population Studies*, 70(1), 1-19.

Chen, X., & Kumar, V. (2024). Machine Learning Applications in Demographic Parameter Estimation. *Journal of Computational Demographics*, 15(2), 78-95.

Duan, Q., et al. (2023). Neural Networks for Population Dynamic Modeling. *Artificial Intelligence Review*, 42(3), 415-434.

Keyfitz, N. (1977). *Applied Mathematical Demography*. New York: Wiley.

Li, N., & Lee, R. (2005). Coherent mortality forecasts for a group of populations: An extension of the Lee-Carter method. *Demography*, 42(3), 575-594.

Murphy, M. (2022). Stochastic Population Forecasting Using Deep Learning. *Demographic Research*, 46, 157-184.

Preston, S. H., Heuveline, P., & Guillot, M. (2000). *Demography: Measuring and Modeling Population Processes*. Blackwell Publishing.

Raftery, A. E., Li, N., Ševčíková, H., Gerland, P., & Heilig, G. K. (2012). Bayesian probabilistic population projections for all countries. *PNAS*, 109(35), 13915-13921.

Raymer, J., & Rogers, A. (2008). Applying and testing principles of mathematical demography. *Population Research and Policy Review*, 27(4), 411-422.

Smith, S. K., & Sincich, T. (2021). Forecasting state and household populations: Applied regression models. *The Annals of Regional Science*, 28(4), 411-426.

Wang, H., & Liu, Y. (2023). Deep Learning Approaches to Demographic Modeling. *Neural Computing and Applications*, 35(8), 5612-5628.

Wilson, T. (2022). Evaluation of simple methods for regional population forecasts. *Population, Space and Place*, 28(2), e2406.

Yang, W., & Zou, J. (2023). Machine Learning in Population Sciences: A Systematic Review. *Artificial Intelligence Review*, 42(1), 145-168.

Zhang, L., et al. (2023). Stochastic Population Models with External Shocks: A Comprehensive Analysis. *Demographic Research*, 48, 233-258.

Zhou, R., & Chen, T. (2024). Parameter Optimization in Demographic Models Using Reinforcement Learning. *Journal of Population Economics*, 37(1), 89-112.