ISSN 2409-2665
Journal of Logistics, Informatics and Service Science
Vol. 8 (2021) No. 1, pp. 133-141
DOI:10.33168/LISS.2021.0109

Population Mean and Extinction Time of a Diffusion Model with Gamma-Distributed Catastrophe Sizes

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Abstract. There is a significant interest in stochastic correspondents of differential equations and classical differences describing occurrences in theoretical models involving population structure. In this paper, the distribution of the extinction time for the linear birth and death diffusion model with Catastrophe is considered. The catastrophes occur at a constant rate, and their magnitudes are random variables having gamma distribution. The population means and the expected time to extinction are also considered for a large initial population size. The particular case when the catastrophe sizes are exponentially distributed is found, too.

Keywords: Diffusion population process, Gamma catastrophe process, Population mean, Extinction time.

1. Introduction

The improvement of mathematical models for population development is of high importance in various fields. For example, the growth and decline of real populations will be approximated in several cases by the solution of a differential equation. However, there are circumstances in which the essentially random nature of population growth should be considered. This leads us to consider stochastic models.

Practically, numerous populations are experiencing sudden catastrophes or large downward jumps. A specific example is the reindeer population of St. Paul Island (Sheffer (1951)). A hard of 25 reindeers was introduced in 1911. By 1938 the number reached approximately 2000, and by 1950 there were was again 8 reindeers only. Another example is the Grizzly bear population in Yellowstone Park (Craighead et al (1974)).

The occurrence of such catastrophes, whether due to a lack of food or disasters such as earthquakes, overflows, epidemics such Pandemic Cofid-19, etc., is mostly analyzed as a random occurrence. However, numerous mathematical models describing the growth of populations subject to catastrophes have been analyzed in detail. In particular, Brockwell (1985) derived the distribution of the time to extinction of a diffusion process with catastrophes occurring at a rate proportional to the population size and having magnitudes with distribution function H (.).

In several different papers, Kaplan et al. (1975), Hanson and Tuckwell (1978), (1981), Pakes et al. (1979), Murthy (1981), Trajstman (1981), Brockwell et al. (1982), (1983), (1985), (1986), Alawneh and Al-Eideh (2000), Al-Eideh and Alawneh (2004) studies have been made of mathematical models for the growth of populations subject to randomly occurring catastrophes from various points of view.

In particular, Brockwell et al. (1982, 1983) consider the class of linear birth and immigration processes subject to catastrophes occurring at a rate proportional to the population size. They examined the distribution of {Xt}, the stationary distribution, and the distribution of time to extinction in the absence of immigration. The analysis was limited to catastrophe sizes having an exponential, uniform or binomial distributions.

Brockwell (1985) found the distribution of the time to extinction for the same model with arbitrary catastrophe size distribution (independent of population size) and also for an analogous diffusion model interrupted by downward jumps with arbitrarily distributed sizes.

Alawneh and Al-Eideh (2000) focuses on an unsolved question related to stochastic model for population growth with catastrophe. More specifically, they consider a diffusion model with catastrophe occurring at a constant rate and having magnitudes with uniform distribution function. The distribution of the extinction time and the asymptotic behavior of the expected time to extinction for large initial population size have been derived.

While, Al-Eideh and Alawneh (2004) determined the exact solution to the Ito differential equations for an Ito diffusion process subject to catastrophe under the assumption that the catastrophe rate is small and their magnitudes are random variables having a general distribution function. The population moments as well as the asymptotic distribution and probabilities of extinction times for large initial population size of such a process are also derived.

In this paper, the distribution of the extinction time for the linear birth and death diffusion model with Catastrophes is considered. the catastrophes occur at a constant rate and their magnitudes are random variables having gamma distribution. The population mean and the expected time to extinction are also considered for large initial population size. The particular case when the catastrophe sizes are exponentially distributed is found, too.

2. Method

Consider the linear birth and death diffusion model in which the diffusion coefficient a and the drift coefficient b are proportional to the population size Xt at time t. The diffusion process is assumed to be interrupted by catastrophes occurring at a constant rate c and having magnitudes with distribution function H(·). Then {X, t ≥0} is a Markov process with state-space S=[0, ∞) and generator g, where

$$\begin{aligned} gf(x) &= axf''(x)+bxf'(x) \\ &\quad + c\int_{[0,x)}[f(x-y)-f(x)]dH_x(y) \\ &\quad + c[f(0)-f(x)][1-H(x^-)], \end{aligned} \tag{1}$$

For all f ∈D(g), where D(g) is the domain of g, and a>0, c>0,. Note that 0 is an absorbing state.

Let F(x, t) be the distribution function of the process at time t, i.e., F(x, t) = P(Xt ≤ x). Let φ(θ, t) be the Laplace transform of Xt, i.e.

$$\varphi(\theta,t)=\int_0^\infty \phi(\theta,x)dF(x,t) \tag{2}$$

Where

$$\phi(\theta,x)=e^{-\theta x} \tag{3}$$

Observe that

$$g\phi = ax\theta^2 e^{-\theta x} - bx\theta e^{-\theta x} + c\int_{[0,x]} [e^{-\theta(x-y)} - e^{-\theta x}]dH_x(y) + c[1-e^{-\theta x}][1-H_x(x^-)] \tag{4}$$

And substituting in the ((Breiman (1968), P. 327))

$$\frac{d}{dt}\int \phi dF(x,t) = \int (g\phi)dF(x,t)$$

Gives

$$\frac{d\varphi}{dt} = \int_0^\infty [ax\theta^2 e^{-\theta x} - bx\theta e^{-\theta x} + c\int_0^{x^-} [e^{-\theta(x-y)} - e^{-\theta x}]dH_x(y) + c[1-e^{-\theta x}][1-H_x(x^-)]]dF(x,t). \tag{5}$$

Since in the section 3, we will consider the Gamma catastrophe distribution Hx(.) which is continuous, then in this case x- may be replaced by x, and by performing the Laplace transforms in (5), we find for the constant catastrophe rate that equation (5) may be written as

$$\frac{d\varphi}{dt} = (b\theta - a\theta^2)\frac{d\varphi}{d\theta} - c\varphi(\theta,t) + c\int_0^\infty e^{-\theta x}\int_0^x e^{\theta y} dH_x(y)dF(x,t) \tag{6}$$

3. Population Mean of a Diffusion Population Model with Constant Gamma Catastrophe Rate

In this section, we consider the population mean *M*(*t*), of a diffusion population process with constant catastrophe rate and Gamma Catastrophe process $H_x(y)$ with parameters (m,λ), $m > 0$, and $\lambda > 0$, defined by

$$H_x(y) = \begin{cases} 0 & , y < 0 \\ \int_0^y \frac{\lambda e^{-\lambda w}(\lambda w)^{m-1}}{\Gamma(m)} dw & , 0 \le y < x, \\ 1 & , y \ge 0 \end{cases} \tag{7}$$

where $\Gamma(m)$, is called the gamma function, is defined as

$$\Gamma(m) = \int_0^\infty e^{-y} y^{m-1} dy$$

Notice that $\Gamma(m) = (m-1)!$ and $\Gamma(1) = 1$ (cf. Ross (1998), P.223).
The equation (6) can now be written as

$$\frac{d\varphi}{dt} = (b\theta - a\theta^2)\frac{d\varphi}{d\theta} - c\varphi(\theta, t) + c\int_0^\infty \varphi(\theta w)\frac{\lambda e^{-\lambda w}(\lambda w)^{m-1}}{\Gamma(m)} dw. \quad (8)$$

Now,

$$\frac{d}{dt}\left(\frac{d\varphi}{d\theta}\right) = (b - 2a\theta)\frac{d\varphi}{d\theta} + (b\theta - a\theta^2)\frac{d^2\varphi}{d\theta^2} - c\frac{d\varphi}{d\theta} + c\int_0^\infty \varphi'(\theta w)\frac{\lambda e^{-\lambda w}(\lambda w)^{m-1}}{\Gamma(m)} dw.$$

By taking $\theta \to 0$, we get

$$\frac{d}{dt}(M(t)) = (b + c\frac{m}{\lambda} - c)M(t) \quad (9)$$

If $X_0 = x$ then the solution of (9) is

$$M(t) = x\exp\left\{\left(b + \frac{cm}{\lambda} - c\right)t\right\} \quad (10)$$

Now, for the special case when Gamma Catastrophe process $H_x(y)$ with parameters (m, λ), where $m = 1$, and $\lambda > 0$, we get

$$H_x(y) = \begin{cases} 0 & , y < 0 \\ 1 - e^{-\lambda y} & , 0 \le y < x, \\ 1 & , y \ge 0 \end{cases} \quad (11)$$

Note that $dH_x(y)$ is Exponential catastrophe process with parameter $\lambda > 0$. Also, using equation (10), the population mean is then given by

$$M(t) = x\exp\left\{\left(b + \frac{c}{\lambda} - c\right)t\right\} \quad (12)$$

4. The Extinction Time of a Diffusion Population Model with Constant Gamma Catastrophe Rate

This section wants to find the distribution of the extinction time and the expected time to extinction for large population size x for the birth-death diffusion model with constant catastrophe rate and beta catastrophe size distribution.

Using the criterion of "drift conditions" (cf. Meyn and Tweedie (1993)) for sure extinction, we can find under what conditions the population will become extinct almost surely. The drift condition is

$$\int yp^1(x,y)dy \le x \tag{13}$$

where $p^1(x,y)$ is the one step transition probability density from the state x to state y.

Note that

$$M(1) = \int yp^1(x,y)dx \tag{14}$$

This implies

$$xe^{\left(b+\frac{cm}{\lambda}-c\right)} \le x \tag{15}$$

After some algebraic manipulations, equation (15) implies

$$b \le \frac{cm}{\lambda} \tag{16}$$

Let q(x) be the probability of eventual extinction given Xo = x is given by

$$q(x)\begin{cases} =1 & if \quad b \le \dfrac{cm}{\lambda} \\ <1 & if \quad b > \dfrac{cm}{\lambda} \end{cases} \tag{17}$$

This means that the population will become extinct almost surely if $b \le cm/\lambda$.

Now, let $T = \inf\{t > 0 : X_t = 0\}$ be the time to extinction and let E_x , denote the expectation conditional on $Xo = x$.

Also, using the "drift conditions" (cf. Meyn and Tweedie (1993)) which says that if

$$\int v(x)p^1(x,y)dy \le v(x) - 1$$

is satisfied for some $v(x)$, then

$$E_xT \le kv(x)$$

where k is constant.

Take $v(x) = x$, we get

$$\int xp^1(x,y)dy \le x-1 \tag{18}$$

this implies that

$$xe^{\left(b+\frac{cm}{\lambda}-c\right)} \le x-1 \tag{19}$$

and then, we get

$$b \le \frac{cm}{\lambda} + \ln\left(1-\frac{1}{x}\right) \tag{20}$$

Now, for large x ($x \to \infty$), equation (20) becomes

$$b \le \frac{cm}{\lambda} \tag{21}$$

Therefore, if $b \le cm/\lambda$ then $E_x T \le kx$ where k is a constant.

For the special case when $dH_x(y)$ is Exponential catastrophe process with parameter $\lambda > 0$. Then, the probability of eventual extinction given *Xo* = *x* is given by

$$q(x)\begin{cases} =1 & if \quad b \le \dfrac{c}{\lambda} \\ <1 & if \quad b > \dfrac{c}{\lambda} \end{cases} \tag{22}$$

and

$$E_x T \le kx \text{ if } \; b \le \frac{c}{\lambda} \text{ as } x \to \infty \,.$$

5. Conclusion

This study provided a methodology for studying the mean and the extinction time of the populations. More specifically, the study departs from the traditional before - and – after regression techniques and the time series analysis and developed a stochastic linear birth and death diffusion model with a constant Gamma catastrophe process that explicitly accounts for the variations in populations in a random environment.

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