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Effect of two imperfect key production subsystems on the optimal number of production cycles

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Abstract. This paper studies the effect of two imperfect key production subsystems (KPS) on the optimal number of production cycles where a product is to be manufactured in batches on a production system over a finite planning horizon. During a production run of the product, the two imperfect KPS may shift from an in-control to an out-of-control state due to three independent sources of shocks. A shock from source 1, 2, and 3 causes the shift on the first KPS, the second KPS, and both KPS's, respectively. Each shocks occur at random time U_1, U_2, and U_3 following exponential distribution with mean $1/\lambda_1$, $1/\lambda_2$, and $1/\lambda_3$, respectively. After each individual shocks, consequently, the production system will produced some defective items with different rates. Three different schemes of defectives rates are considered here which are constant, linear increasing function of time, and exponential increasing function of time. For each scheme, a mathematical model describing the situation is developed in order to determine a number of production cycles that minimizes the expected total cost per unit time including setup, inventory carrying, and defective costs. Solution approach of finding optimal solution of the models is provided, together with some numerical examples.

Keywords: Production lot sizing, Economic production quantity, Finite planning horizon, Imperfect production system, Shock model

1. Introduction

The problem considered here is a variant of classical Economic Production Quantity (EPQ) problem where a product is to be manufactured in batches on an imperfect production system over a finite planning horizon, in which the production system consists of two imperfect key production subsystems (KPS). This work has twofold motivations that are practical and theoretical aspects. In the practical aspect, it is common that the production system is imperfect and has two key production systems in it. In addition, the product life cycle tends to be shorter. Therefore, it is important to address the EPQ on imperfect production system over finite planning horizon problem.

In the theoretical aspect, although some EPQ on imperfect production system models have been proposed before, this problem never been discussed before. Most of the models considered production system with single KPS only. The initial work of Rosenblatt and Lee (1986b) developed an EPQ model with an imperfect production process where the production process may shift at a random point in time from an in-control state to an out-of-control state during a production run, in which the time-to-shift is an exponentially distributed random variable and a fixed portion of items produced become defective after the process has shifted. After this work, several models dealt with various additional system setting, although still subject to single KPS only. Since the model proposed in this paper can be viewed as an extension of their model, a review of results related to Rosenblatt and Lee's model is provided below.

Inspection schedule is one of the additional system setting that was included in the model. Lee and Rosenblatt (1987) started to consider it by including a periodic inspection schedules into the model proposed in Rosenblatt and Lee (1986b). Furthermore, Rosenblatt and Lee (1986a) compared a continuous inspection schedule and a periodic inspection schedule. After that, Lee and Rosenblatt (1988) investigated different policies of providing process inspection and restoration capabilities in reducing the defective cost. Later, Lee and Rosenblatt (1989) studied a model with an equal-spaced inspection schedule and a restoration cost as a function of detection delay. Additional system settings related to inspection that had been considered are the reworking cost before sale and the warranty cost after sale (Lee & Park, 1991), increasing failure rate function (Lin, Tseng & Liou, 1991), impact of inspection errors (Liou, Tseng & Lin, 1994; Ben-Daya, 1999), inspection times that is discrete or continuous (Makis, 1998), partial inspection policy (Wang & Sheu, 2001), and the joint effects of maintenance policy by inspection and the production-inventory system, including raw materials on the cost of operating a single facility (Lin, Chen & Kroll, 2003).

Various EPQ models of imperfect production system proposed in the past also included a range of time-to-shift distribution. While Rosenblatt and Lee (1986b) used exponential time-to-shift distribution, many other research used different distribution such as a function of production rate (Khouja & Mehrez, 1994), general distribution (Hariga & Ben-Daya, 1998; Giri & Dohi, 2007), and arbitrary distribution (Kim & Hong, 1999).

The others additional additional system setting that was included in the EPQ models of imperfect production system are backorders and variable leadtime (Ouyang & Chang, 2000), allowable shortage (Chung & Hou, 2003), and standby key modules (Hsieh & Lee, 2005).

Recent research in this area are filled with more complex problem settings and to be integrated with other area of production system. Hou and Lin (2005) studied a model where defective items are also generated even the production process is in the in-control state. Ben-Daya, Noman, and Hariga (2006) investigated integrated inventory inspection models with and without replacement of nonconforming items with a deterministic demand. Lin and Lin (2007) developed models for an imperfect production system in order to deal with situations regarding whether and when a screening process is implemented with scraps and no shortages. Huang, Lo, and Ho (2008) designed an effective inspection schema for imperfect production systems. Ben-Daya and Noman (2008) developed integrated inventory inspection models with and without replacement of nonconforming items for stochastic demand. Chakraborty, Giri, and Chaudhuri (2009) investigated the joint effect of a process shift, a machine breakdown and inspections on lot sizing decisions for an imperfect production system under two inspection policies. Wang and Tsai (2012) proposed a heuristic inspection policy for materials and products. They first obtained the inspection range for the input material without considering product inspection, and then determined the product inspection range based on the obtained range of the input material inspection.

In contrast with previous models which is considered production system with single KPS only, Lin and Gong (2011) recently studied an EPQ model where the production system is imperfect and dictated by two imperfect KPS's over an infinite planning horizon. They proposed a model that minimizes the total production cost including cost of setup, inventory carrying, and quality. In their model, the two imperfect KPS may shift from an in-control to an out-of-control state due to three independent sources of shocks. When at least one KPS on out-of-control state, consequently, the production system will produce some defective items with fixed but different rates. Their model then can be considered as an extension of Rosenblatt and Lee (1986b) model in terms of number of KPS. Ai et al. (2013) proposed three approaches for solving the fixed defective rates model. In this paper, instead of infinite planning horizon, we consider the finite planning horizon. Furthermore, the model is extended to two cases where defective rates become a linear function and an exponential function of time.

In the next section, we propose a basic mathematical model where the defective rates are constant after the production system has shifted into various out-of-control states. A solution approach is developed to obtain a optimal solution of the number of production cycles. Then, the basic model is extended to two cases where defective rates become a linear function or an exponential function of time. Numerical examples and results are also provided to illustrate the effectiveness of the proposed solution approach. Finally, concluding remarks are given along with some future research directions.

2. Problem Definition and Mathematical Model

The basic problem considered here is a variant of classical Economic Production Quantity (EPQ) problem where a product is to be manufactured in batches on an imperfect production system over a finite planning horizon (H). The demand rate of the product (d) is constant. During a production run of the product at the constant rate p, the machine is dictated by two unreliable key production subsystems (KPS's). We assume that at the beginning of a production run, the production system is in an in-control state. We will name it state 0. Furthermore, the production system is subject to three independent sources of shocks. A shock from source 1 causes the first KPS to shift into an out-of-control state. We will name it state 1. In this state, only the first KPS has shifted into the out-of-control state. As a result, a fixed α percentage of defective items are produced. The cost incurred by producing a defective item when the production system is in state 1 is π_1 which could represent the cost or rework, repair, replacement, or loss of goodwill. The shock occurs at a random time U_1 following an exponential distribution with a mean $1/\lambda_1$. On the other hand, a shock from source 2 causes the second KPS to shift into an out-of-control state. It is named state 2. In this state, only the second KPS has shifted into the out-of-control state. Consequently, a fixed β percentage of defective items are produced, and the cost incurred by producing such a unit is π_2. It occurs at a random time U_2 following an exponential distribution with a mean

$1/\lambda_2$. Finally, a shock from source 3 will result in both KPS's to shift into an out-of-control state. It is named state 3. This will cause a fixed δ percentage of defective items to be produced. The cost incurred by producing a defective item in this state is π_3 . This shock occurs at a random time U_3 following an exponential distribution with a mean $1/\lambda_3$.

The optimization problem is to determining optimal number of production cycles *n*, that minimizes the expected total cost per unit time including setup, inventory carrying, and defective costs. The expected total cost, *Z*(*n*), can be expressed as follow:

$$Z(n) = n\left[A + h\overline{I}(n) + \pi_1 E[N_1(n)] + \pi_2 E[N_2(n)] + \pi_3 E[N_3(n)] \right] \tag{1}$$

where A is the setup cost, h is the holding cost, $\overline{I}(n)$ is the total inventory per production cycle, $E[N_i(n)]$ is the expected number of defective items during a production cycle when the production system in state *i*.

As developed by Lin and Gong (2011), the expected number of defective items during a production cycle when the production system in state 1, 2, 3 can be expressed as Equations (2), (3), (4), respectively:

$$E[N_1(\tau)] = p\alpha\left(\frac{1-\exp[-(\lambda_2+\lambda_3)\tau]}{\lambda_2+\lambda_3} - \frac{1-\exp[-(\lambda_1+\lambda_2+\lambda_3)\tau]}{\lambda_1+\lambda_2+\lambda_3} \right) \tag{2}$$

$$E[N_2(\tau)] = p\beta\left(\frac{1-\exp[-(\lambda_1+\lambda_3)\tau]}{\lambda_1+\lambda_3} - \frac{1-\exp[-(\lambda_1+\lambda_2+\lambda_3)\tau]}{\lambda_1+\lambda_2+\lambda_3} \right) \tag{3}$$

$$E[N_3(\tau)] = p\delta\left(\frac{\exp[-(\lambda_1+\lambda_3)\tau]+(\lambda_1+\lambda_3)\tau-1}{\lambda_1+\lambda_3} + \frac{\exp[-(\lambda_2+\lambda_3)\tau]+(\lambda_2+\lambda_3)\tau-1}{\lambda_2+\lambda_3} - \frac{\exp[-(\lambda_1+\lambda_2+\lambda_3)\tau]+(\lambda_1+\lambda_2+\lambda_3)\tau-1}{\lambda_1+\lambda_2+\lambda_3} \right) \tag{4}$$

where τ is the production uptime.

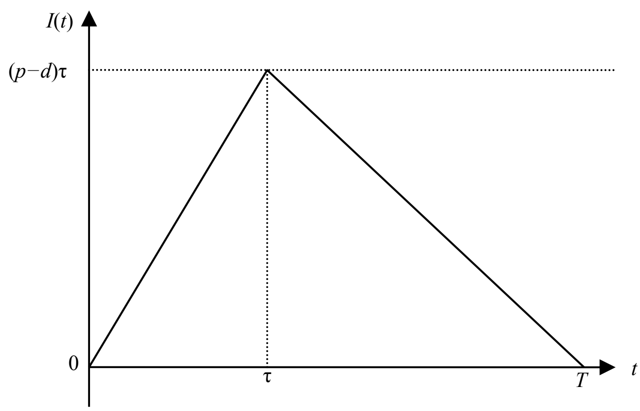

Fig. 1: A Production Cycle.

Since the inventory level in a production cycle can be illustrated as Figure 1, with a finite planning horizon H and pre-determined number of production cycles n, the length of a production cycle can be written as

$$T = H/n \tag{5}$$

Therefore, the production uptime can be expressed as

$$\tau = dH/pn \tag{6}$$

Substituting equation (6) to equations (2) – (4), we obtained

$$E[N_1(n)] = p\alpha\left(\frac{1-\exp\left[-\left(\lambda_2+\lambda_3\right)\frac{dH}{pn}\right]}{\lambda_2+\lambda_3} - \frac{1-\exp\left[-\left(\lambda_1+\lambda_2+\lambda_3\right)\frac{dH}{pn}\right]}{\lambda_1+\lambda_2+\lambda_3}\right) \tag{7}$$

$$E[N_2(n)] = p\beta\left(\frac{1-\exp\left[-\left(\lambda_1+\lambda_3\right)\frac{dH}{pn}\right]}{\lambda_1+\lambda_3} - \frac{1-\exp\left[-\left(\lambda_1+\lambda_2+\lambda_3\right)\frac{dH}{pn}\right]}{\lambda_1+\lambda_2+\lambda_3}\right) \tag{8}$$

$$\begin{aligned} E[N_3(n)] = p\delta\Bigg(& \frac{\exp\left[-\left(\lambda_1+\lambda_3\right)\frac{dH}{pn}\right]+\left(\lambda_1+\lambda_3\right)\frac{dH}{pn}-1}{\lambda_1+\lambda_3} + \frac{\exp\left[-\left(\lambda_2+\lambda_3\right)\frac{dH}{pn}\right]}{\lambda_2+\lambda_3} + \\ & \frac{\left(\lambda_2+\lambda_3\right)\frac{dH}{pn}-1}{\lambda_2+\lambda_3} - \frac{\exp\left[-\left(\lambda_1+\lambda_2+\lambda_3\right)\frac{dH}{pn}\right]+\left(\lambda_1+\lambda_2+\lambda_3\right)\frac{dH}{pn}-1}{\lambda_1+\lambda_2+\lambda_3}\Bigg) \end{aligned} \tag{9}$$

The total inventory per production cycle can be expressed as

$$\bar{I}(n) = \frac{1}{2}\frac{H^2}{n^2}(p-d)\frac{d}{p} \tag{10}$$

Substituting equation (10) to (1), therefore, the objective function can be expressed as

$$Z(n) = n\left\{A + h\frac{1}{2}\frac{H^2}{n^2}(p-d)\frac{d}{p} + \pi_1 E\left[N_1(n)\right] + \pi_2 E\left[N_2(n)\right] + \pi_3 E\left[N_3(n)\right]\right\} \tag{11}$$

3. Solution Methodology for the Basic Model

The basic mathematical problem formulated in Section 2, as seen in equation (11), is presented in the form of single objective optimization with positive integer decision variable, which is n. Although complete enumeration of the decision variable can be used for finding the optimal solution, we develop an algorithm for finding the optimal solution with smaller computational effort.

The algorithm is developed based on MacLaurin approximation on the objective function. We apply MacLaurin series up to third order to approximate any exponential function in the objective function, i.e. by following equation:

$$\exp(-\lambda\tau) \approx 1 - \lambda\tau + \frac{1!}{2}(\lambda\tau)^2 - \frac{1}{3!}(\lambda\tau)^3 \tag{12}$$

Substituting equation (12) to equations (7) – (9) and after some algebra, we obtain:

$$E[N_1(n)] \approx p\alpha\left[\frac{1}{2}\lambda_1\frac{d^2H^2}{p^2}\frac{1}{n^2}-\frac{1}{6}\lambda_1(\lambda_1+2\lambda_2+2\lambda_3)\frac{d^3H^3}{p^3}\frac{1}{n^3}\right] \tag{13}$$

$$E[N_2(n)] \approx p\beta\left[\frac{1}{2}\lambda_2\frac{d^2H^2}{p^2}\frac{1}{n^2}-\frac{1}{6}\lambda_2(2\lambda_1+\lambda_2+2\lambda_3)\frac{d^3H^3}{p^3}\frac{1}{n^3}\right] \tag{14}$$

$$E[N_3(n)] \approx p\delta\left[\frac{1}{2}\lambda_3\frac{d^2H^2}{p^2}\frac{1}{n^2}-\frac{1}{6}(\lambda_3{}^2-2\lambda_1\lambda_2)\frac{d^3H^3}{p^3}\frac{1}{n^3}\right] \tag{15}$$

Therefore, the objective function can be approximated as

$$\begin{aligned}\tilde{Z}(n) = n\Bigg\{ & A+\frac{hH^2(p-d)}{2n^2}\frac{d}{p}+\pi_1 p\alpha\left[\frac{1}{2}\lambda_1\frac{d^2H^2}{p^2}\frac{1}{n^2}-\frac{1}{6}\lambda_1(\lambda_1+2\lambda_2+2\lambda_3)\frac{d^3H^3}{p^3}\frac{1}{n^3}\right] \\ & +\pi_2 p\beta\left[\frac{1}{2}\lambda_2\frac{d^2H^2}{p^2}\frac{1}{n^2}-\frac{1}{6}\lambda_2(2\lambda_1+\lambda_2+2\lambda_3)\frac{d^3H^3}{p^3}\frac{1}{n^3}\right] \\ & +\pi_3 p\delta\left[\frac{1}{2}\lambda_3\frac{d^2H^2}{p^2}\frac{1}{n^2}-\frac{1}{6}(\lambda_3{}^2-2\lambda_1\lambda_2)\frac{d^3H^3}{p^3}\frac{1}{n^3}\right] \Bigg\}\end{aligned} \tag{16}$$

or can be stated as

$$\tilde{Z}(n) = nA+\frac{1}{n}B-\frac{1}{n^2}C \tag{17}$$

where

$$B=\frac{H^2}{2}(p-d)\frac{d}{p}h+\frac{d^2H^2}{p}\left[\pi_1\alpha\frac{1}{2}\lambda_1+\pi_2\beta\frac{1}{2}\lambda_2+\pi_3\delta\frac{1}{2}\lambda_3\right] \tag{18}$$

$$C=\frac{d^3H^3}{6p^2}\left[\pi_1\alpha\lambda_1(\lambda_1+2\lambda_2+2\lambda_3)+\pi_2\beta\lambda_2(2\lambda_1+\lambda_2+2\lambda_3)+\pi_3\delta\left(\lambda_3{}^2-2\lambda_1\lambda_2\right)\right] \tag{19}$$

In order to develop proposed algorithm, we derive two properties of equation (17), the approximation function of the objective. Property 3.1 shows that $\tilde{Z}(n)$ is a convex function for certain conditions and Property 3.2 show the conditions for obtaining n^*, the optimal number of production cycles.

Property 3.1. $\tilde{Z}(n)$ *is a convex function of n when* $C \neq 0$ *and* $n \geq 3C/B$

Proof.

A function is a convex function whenever the second derivative of that function is greater or equal to zero. Applying this condition for $\tilde{Z}(n)$, we found that

$$\tilde{Z}'(n) = A-\frac{B}{n^2}+\frac{2C}{n^3}$$

$$\tilde{Z}''(n) = \frac{2B}{n^3}-\frac{6C}{n^4} \geq 0$$

$$\Leftrightarrow \quad n \geq \frac{3C}{B} \qquad \square$$

Property 3.2. *The minimum value of* $\tilde{Z}(n)$ *is obtained at* $n^*>1$ *that satisfied the following conditions:*

$$A>\frac{B}{n^*(n^*+1)}-\frac{(2n^*+1)}{n^{*2}(n^*+1)^2}C \text{ and } A<\frac{B}{n^*(n^*-1)}-\frac{(2n^*-1)}{n^{*2}(n^*-1)^2}C$$

Proof.

At the minimum point, the objective function value $\tilde{Z}(n)$ must be smaller than its adjacent points. Therefore, $\tilde{Z}(n^*)<\tilde{Z}(n^*+1)$ and $\tilde{Z}(n^*)<\tilde{Z}(n^*-1)$.

The first condition can be simplified into

$$\tilde{Z}(n^*)<\tilde{Z}(n^*+1)$$

$$n^*A+\frac{1}{n^*}B-\frac{1}{n^{*2}}C<(n^*+1)A+\frac{1}{(n^*+1)}B-\frac{1}{(n^*+1)^2}C$$

$$(n^*+1-n^*)A>\left[\frac{1}{n^*}-\frac{1}{(n^*+1)}\right]B-\left[\frac{1}{n^{*2}}-\frac{1}{(n^*+1)^2}\right]C$$

$$A>\frac{B}{n^*(n^*+1)}-\frac{2n^*+1}{n^{*2}(n^*+1)^2}C$$

The second condition can be simplified into

$$\tilde{Z}(n^*)<\tilde{Z}(n^*-1)$$

$$n^*A+\frac{1}{n^*}B-\frac{1}{n^{*2}}C<(n^*-1)A+\frac{1}{(n^*-1)}B-\frac{1}{(n^*-1)^2}C$$

$$(n^*-n+1)A<\left[\frac{1}{(n^*-1)}-\frac{1}{n^*}\right]B-\left[\frac{1}{(n^*-1)^2}-\frac{1}{n^{*2}}\right]C$$

$$A<\frac{B}{n^*(n^*-1)}-\frac{2n^*-1}{n^{*2}(n^*-1)^2}C$$

□

Based on the two properties above, we can propose algorithm 1 for finding optimal number of production cycles (n^*). It is noted that this optimal value is derived from the approximation of the objective function ($\tilde{Z}(n)$), but we are expecting that the result is also apply for the actual objective function ($Z(n)$).

Algorithm 1: Algorithm for Obtaining the Optimal Value of *n**

Step 0. Calculate B and C from equations (18) and (19), respectively. Set $n=\lceil 3C/B \rceil$. If $n>1$, go to Step 1. If $n=1$ and $\tilde{Z}(n)<\tilde{Z}(n+1)$, set the optimal value of $n^*=1$ and Stop, otherwise set $n=n+1$ and go to Step 1.

Step 1. Calculate $\phi_U = \frac{B}{n(n+1)} - \frac{(2n+1)}{n^2(n+1)^2}C$ and $\phi_L = \frac{B}{n(n-1)} - \frac{(2n-1)}{n^2(n-1)^2}C$

Step 2. If $A > \phi_U$ and $A < \phi_L$ set the optimal value of $n^* = n$ and Stop, otherwise set $n = n+1$ and return to Step 1.

4. Numerical Examples

Three cases with different problem parameters are presented below in order to show the generality of the proposed algorithm.

4.1. Example Case 1

Consider the problem with following parameters: p = 300, d = 200, A = 30, h = 0.08, $\pi_1 = 10$, $\pi_2 = 10$, $\pi_3 = 12$, $\alpha = 0.1$, $\beta = 0.1$, $\delta = 0.16$, $\lambda_1 = 0.05$, $\lambda_2 = 0.1$, $\lambda_3 = 0.2$, $H = 2$.

Following the Algorithm 1:

Step 0. After calculation, the value of B = 60.9067 and C = 2.3784. Therefore, n = 1. Since $\tilde{Z}(n) = \tilde{Z}(1) = 88.5282$ and $\tilde{Z}(n+1) = \tilde{Z}(2) = 89.8587$, $\tilde{Z}(n) < \tilde{Z}(n+1)$, therefore the optimal value of $n^* = 1$. Stop

In order to verify the result from Algorithm 1, following table shows the objective function value of $Z(n)$ which are calculated using equation (11). This table confirms that the minimum cost is found at n = 1, showing that this algorithm is able to find the optimum value of this case.

Table 1: The Objective Function Value of for Example Case 1

n	$Z(n)$
1	*88.6162*
2	89.8699
3	110.0412
4	135.0794
5	162.0869

4.2. Example Case 2

Consider the problem with following parameters: p = 300, d = 200, A = 100, h = 0.08, $\pi_1 = 10$, $\pi_2 = 10$, $\pi_3 = 12$, $\alpha = 0.1$, $\beta = 0.1$, $\delta = 0.16$, $\lambda_1 = 0.05$, $\lambda_2 = 0.1$, $\lambda_3 = 0.2$, $H = 10$.

Following the Algorithm 1:

Step 0. After calculation, the value of B = 1522.6667 and C = 297.3037. Therefore, n = 1. Since $\tilde{Z}(n) = \tilde{Z}(1) = 1325.3630$ and

$\tilde{Z}(n+1)=\tilde{Z}(2)=887.0074$, $\tilde{Z}(n)>\tilde{Z}(n+1)$, $n=n+1=2$ and go to Step 1.
Step 1 and *Step 2*. Iteration of Step 1 and Step 2 are presented in Table 2, and finally found the optimal value of $n^*=4$. Stop.

Table 2: Iteration of Step 1 and Step 2 for Example Case 2

n	ϕ_U	ϕ_L	$A<\phi_L$ and $A>\phi_U$
2	212.4856	538.3556	No
3	112.4366	212.4856	No
4	69.4440	112.4366	Yes

In order to verify the result from Algorithm 1 for this case, Table 3 shows the objective function value of $Z(n)$ which are calculated using equation (11). This table confirms that the minimum cost is found at n = 4, showing that this algorithm is also able to find the optimum value of this case.

Table 3: The Objective Function Value of for Example Case 2

n	$Z(n)$
1	1374.0653
2	893.5641
3	776.5151
4	*762.9372*
5	793.0809
6	845.7751

4.3. Example Case 3

Consider the problem with following parameters: p = 300, d = 200, A = 100, h = 0.08, $\pi_1=10$, $\pi_2=10$, $\pi_3=12$, $\alpha=0.1$, $\beta=0.1$, $\delta=0.16$, $\lambda_1=0.25$, $\lambda_2=0.5$, $\lambda_3=0.1$, $H=10$.

Following the Algorithm 1:
Step 0. After calculation, the value of B = 6546.6667 and C = 7432.5926. Therefore, n = 4 and go to Step 1.
Step 1 and *Step 2*. Iteration of Step 1 and Step 2 are presented in Table 4, and finally found the optimal value of $n^*=7$. Stop.

Table 4: Iteration of Step 1 and Step 2 for Example Case 3

n	ϕ_U	ϕ_L	$A<\phi_L$ and $A>\phi_U$
4	160.1000	184.2490	No
5	127.3794	160.1000	No
6	101.0977	127.3794	No
7	81.3535	101.0977	Yes

In order to verify the result from Algorithm 1 for this case, Table 5 shows the objective function value of $Z(n)$ which are calculated using equation (11). This table confirms that the minimum cost is found at n = 7, showing that this algorithm is consistently able to find the optimum value of this case.

Table 5: The Objective Function Value of for Example Case 3

n	$Z(n)$
4	1663.931
5	1560.732
6	1513.526
7	*1502.060*
8	1514.765
9	1544.565

5. Linear and Exponential Deteriorating Rates

In this section, we consider two cases where once a KPS has shifted to the out-of-control state, the percentage of defective items increases as the production system deteriorates over time. The first case is the problem with linear deteriorating rates, and the second case is the problem with exponential deteriorating rates.

5.1. Linear Deteriorating Rates

In the first case, at the beginning of a production run, only defect-free products are manufactured. As time goes on, either the first KPS or the second KPS or both may shift from an in-control state to an out-of-control state. If the first KPS has shifted at time x before the production run time τ is reached while the second KPS is still in the in-control state, then the production system is in state 1 and will begin to deteriorate linearly. As a result, defective products are generated at a rate of $\alpha_0+\alpha_1(t-x)$, $x<t<\tau$, $\alpha_0>0$ and $\alpha_1>0$, at any time t after the shift. Furthermore, if the second KPS has also shifted at time y before

τ is reached, then the production system is in state 3 and defective products are produced at a rate of $\delta_0 + \delta_1(t-y)$, $x<y<t<\tau$, $\delta_0>0$ and $\delta_1>0$, at any time *t* after the shift. By the same token, if the second KPS has shifted at time *y* while the first KPS is still in the in-control state, then the production system is in state 2 and will begin to produce defective products at a rate of $\beta_0+\beta_1(t-y)$, $y<t<\tau$, $\beta_0>0$ and $\beta_1>0$, at any time *t* after the shift.

In this case, the number of defective items produced during a production run can be expressed as

$$\begin{aligned} E[N_1(\tau)] = {} & p\int_0^{\tau}\int_0^{y}[\alpha_0(y-x)]f_{X,Y}(x,y)dxdy + p\int_0^{\tau}\int_0^{y}\left[\int_0^{y-x}\alpha_1 tdt\right]f_{X,Y}(x,y)dxdy \\ & + p\int_{\tau}^{\infty}\int_0^{\tau}[\alpha_0(\tau-x)]f_{X,Y}(x,y)dxdy + p\int_{\tau}^{\infty}\int_0^{\tau}\left[\int_0^{\tau-x}\alpha_1 tdt\right]f_{X,Y}(x,y)dxdy \end{aligned} \quad (20)$$

$$\begin{aligned} E[N_2(\tau)] = {} & p\int_0^{\tau}\int_0^{x}[\beta_0(x-y)]f_{X,Y}(x,y)dydx + p\int_0^{\tau}\int_0^{x}\left[\int_0^{x-y}\beta_1 tdt\right]f_{X,Y}(x,y)dydx \\ & + p\int_{\tau}^{\infty}\int_0^{\tau}[\beta_0(\tau-y)]f_{X,Y}(x,y)dydx + p\int_{\tau}^{\infty}\int_0^{\tau}\left[\int_0^{\tau-y}\beta_1 tdt\right]f_{X,Y}(x,y)dydx \end{aligned} \quad (21)$$

$$\begin{aligned} E[N_3(\tau)] = {} & p\int_0^{\tau}\int_0^{x}[\delta_0(\tau-x)]f_{X,Y}(x,y)dydx + p\int_0^{\tau}\int_0^{x}\left[\int_0^{\tau-x}\delta_1 tdt\right]f_{X,Y}(x,y)dydx \\ & + p\int_0^{\tau}\int_0^{y}[\delta_0(\tau-y)]f_{X,Y}(x,y)dxdy + p\int_0^{\tau}\int_0^{y}\left[\int_0^{\tau-y}\delta_1 tdt\right]f_{X,Y}(x,y)dxdy \\ & + p\int_0^{\tau}\delta_0(\tau-x)f_{X,Y}(x,y)dxdy + p\int_0^{\tau}\left[\int_0^{\tau-x}\delta_1 tdt\right]f_{X,Y}(x,y)dxdy \end{aligned} \quad (22)$$

where $f_{X,Y}(x,y)$ is the joint probability density function that is expressed as following:

$$f_{X,Y}(x,y)=\begin{cases} \lambda_2(\lambda_1+\lambda_3)\exp[-\lambda_2 y-(\lambda_1+\lambda_3)x], & if\ (X,Y)\in\{0\le Y<X\le\tau\} \\ \lambda_1(\lambda_2+\lambda_3)\exp[-\lambda_1 x-(\lambda_2+\lambda_3)y], & if\ (X,Y)\in\{0\le X<Y\le\tau\} \\ \lambda_2(\lambda_1+\lambda_3)\exp[-\lambda_2 y-(\lambda_1+\lambda_3)x], & if\ (X,Y)\in\{0\le Y\le\tau\le X\} \\ \lambda_1(\lambda_2+\lambda_3)\exp[-\lambda_1 x-(\lambda_2+\lambda_3)y], & if\ (X,Y)\in\{0\le X\le\tau\le Y\} \\ \lambda_3\exp[-(\lambda_1+\lambda_2+\lambda_3)x] & if\ (X,Y)\in\{0\le X=Y\le\tau\} \end{cases} \quad (23)$$

After doing the integrations and substituting equation 6, we obtain

$$E[N_1(n)] = p\alpha_0\left(\frac{1-\exp\left[-(\lambda_2+\lambda_3)\frac{dH}{pn}\right]}{\lambda_2+\lambda_3} - \frac{1-\exp\left[-(\lambda_1+\lambda_2+\lambda_3)\frac{dH}{pn}\right]}{\lambda_1+\lambda_2+\lambda_3}\right) + p\alpha_1\left(\frac{1-\exp\left[-(\lambda_2+\lambda_3)\frac{dH}{pn}\right]-(\lambda_2+\lambda_3)\frac{dH}{pn}\exp\left[-(\lambda_2+\lambda_3)\frac{dH}{pn}\right]}{(\lambda_2+\lambda_3)^2} + \frac{1-\exp\left[-(\lambda_1+\lambda_2+\lambda_3)\frac{dH}{pn}\right]}{\lambda_1(\lambda_1+\lambda_2+\lambda_3)} - \frac{1-\exp\left[-(\lambda_2+\lambda_3)\frac{dH}{pn}\right]}{\lambda_1(\lambda_2+\lambda_3)}\right) \quad (24)$$

$$E[N_2(n)] = p\beta_0\left(\frac{1-\exp\left[-(\lambda_1+\lambda_3)\frac{dH}{pn}\right]}{\lambda_1+\lambda_3} - \frac{1-\exp\left[-(\lambda_1+\lambda_2+\lambda_3)\frac{dH}{pn}\right]}{\lambda_1+\lambda_2+\lambda_3}\right) + p\beta_1\left(\frac{1-\exp\left[-(\lambda_1+\lambda_3)\frac{dH}{pn}\right]-(\lambda_1+\lambda_3)\frac{dH}{pn}\exp\left[-(\lambda_1+\lambda_3)\frac{dH}{pn}\right]}{(\lambda_1+\lambda_3)^2} + \frac{1-\exp\left[-(\lambda_1+\lambda_2+\lambda_3)\frac{dH}{pn}\right]}{\lambda_2(\lambda_1+\lambda_2+\lambda_3)} - \frac{1-\exp\left[-(\lambda_1+\lambda_3)\frac{dH}{pn}\right]}{\lambda_2(\lambda_1+\lambda_3)}\right) \quad (25)$$

$$E[N_3(n)] = p\delta_0\left(\frac{\exp\left[-(\lambda_1+\lambda_3)\frac{dH}{pn}\right]+(\lambda_1+\lambda_3)\frac{dH}{pn}-1}{\lambda_1+\lambda_3} + \frac{\exp\left[-(\lambda_2+\lambda_3)\frac{dH}{pn}\right]+(\lambda_2+\lambda_3)\frac{dH}{pn}-1}{\lambda_2+\lambda_3} - \frac{\exp\left[-(\lambda_1+\lambda_2+\lambda_3)\frac{dH}{pn}\right]+(\lambda_1+\lambda_2+\lambda_3)\frac{dH}{pn}-1}{\lambda_1+\lambda_2+\lambda_3}\right) + p\delta_1\left[\frac{\left(\frac{dH}{pn}\right)^2}{2} - \frac{1-\exp\left[-(\lambda_1+\lambda_2+\lambda_3)\frac{dH}{pn}\right]-(\lambda_1+\lambda_2+\lambda_3)\frac{dH}{pn}}{(\lambda_1+\lambda_2+\lambda_3)^2} + \frac{1-\exp\left[-(\lambda_1+\lambda_3)\frac{dH}{pn}\right]-(\lambda_1+\lambda_3)\frac{dH}{pn}}{(\lambda_1+\lambda_3)^2} + \frac{1-\exp\left[-(\lambda_2+\lambda_3)\frac{dH}{pn}\right]-(\lambda_2+\lambda_3)\frac{dH}{pn}}{(\lambda_2+\lambda_3)^2}\right] \quad (26)$$

Therefore the objective function of linear deteriorating rates model is the equation (11) with updated expression on $E[N_1(n)]$, $E[N_2(n)]$, and $E[N_3(n)]$, which are following equations (24), (25), (26), respectively.

Similar to the basic model, after applying Maclaurin series in the equation (12) to approximate every exponential function in the expression, and doing some algebra, we can obtain the approximation of the objective function as follow

$$\tilde{Z}(n) = nA + \frac{1}{n}B_L - \frac{1}{n^2}C_L \quad (27)$$

where

$$B_L = \frac{H^2}{2}\frac{d^2}{p}\left[\frac{h}{d}(p-d) + \pi_1\alpha_0\lambda_1 + \pi_2\beta_0\lambda_2 + \pi_3\delta_0\lambda_3\right] \quad (28)$$

$$C_L = \frac{d^3H^3}{6p^2}\Big(\pi_1\lambda_1\left[(\lambda_1+2\lambda_2+2\lambda_3)\alpha_0-\alpha_1\right]+\pi_2\lambda_2\left[(2\lambda_1+\lambda_2+2\lambda_3)\beta_0-\beta_1\right] + \pi_3\lambda_3\left[\left(\lambda_3{}^2-2\lambda_1\lambda_2\right)\delta_0-\delta_1\right]\Big) \tag{29}$$

Since the approximation of the objective function of the linear deteriorating rates model is in the same form as its of the basic model, the solution approach developed in Section 3 can be utilized to find the optimal solution with B_L and C_L replacing B and C appearing in the Algorithm 1.

5.2. Exponential Deteriorating Rates

In this second case, we consider a case where, if the first KPS, or the second KPS, or both KPSs are shifted from an in-control state to an out-of-control state, defective items produced will be generated at a rate of $\alpha_0+\alpha_1\left[1-\exp\left(-\alpha_2 t\right)\right]$, $\beta_0+\beta_1\left[1-\exp\left(-\beta_2 t\right)\right]$, and $\delta_0+\delta_1\left[1-\exp\left(-\delta_2 t\right)\right]$, respectively. The expected number of defective items produced during a production run can be expressed as follows:

$$\begin{aligned} E[N_1(\tau)] &= p\int_0^\tau\int_0^y[\alpha_0(y-x)]f_{X,Y}(x,y)dxdy + p\int_0^\tau\int_0^y\left[\int_0^{y-x}\alpha_1\left(1-\exp(-\alpha_2 t)\right)dt\right]f_{X,Y}(x,y)dxdy \\ &+p\int_\tau^\infty\int_0^\tau[\alpha_0(\tau-x)]f_{X,Y}(x,y)dxdy + p\int_\tau^\infty\int_0^\tau\left[\int_0^{\tau-x}\alpha_1\left(1-\exp(-\alpha_2 t)\right)dt\right]f_{X,Y}(x,y)dxdy \end{aligned} \tag{30}$$

$$\begin{aligned} E[N_2(\tau)] &= p\int_0^\tau\int_0^x[\beta_0(x-y)]f_{X,Y}(x,y)dydx + p\int_0^\tau\int_0^x\left[\int_0^{x-y}\beta_1\left(1-\exp(-\beta_2 t)\right)dt\right]f_{X,Y}(x,y)dydx \\ &+p\int_\tau^\infty\int_0^\tau[\beta_0(\tau-y)]f_{X,Y}(x,y)dydx + p\int_\tau^\infty\int_0^\tau\left[\int_0^{\tau-y}\beta_1\left(1-\exp(-\beta_2 t)\right)dt\right]f_{X,Y}(x,y)dydx \end{aligned} \tag{31}$$

$$\begin{aligned} E[N_3(\tau)] &= p\int_0^\tau\int_0^x[\delta_0(\tau-x)]f_{X,Y}(x,y)dydx + p\int_0^\tau\int_0^x\left[\int_0^{\tau-x}\delta_1\left(1-\exp(-\delta_2 t)\right)dt\right]f_{X,Y}(x,y)dydx \\ &+p\int_0^\tau\int_0^y[\delta_0(\tau-y)]f_{X,Y}(x,y)dxdy + p\int_0^\tau\int_0^y\left[\int_0^{\tau-y}\delta_1\left(1-\exp(-\delta_2 t)\right)dt\right]f_{X,Y}(x,y)dxdy \\ &+p\int_0^\tau\delta_0(\tau-x)f_{X,Y}(x,y)dxdy + p\int_0^\tau\left[\int_0^{\tau-x}\delta_1\left(1-\exp(-\delta_2 t)\right)dt\right]f_{X,Y}(x,y)dxdy \end{aligned} \tag{32}$$

Following the similar procedure with the basic and the linear deteriorating rates model, we obtain the approximation of the objective function as follow

$$\tilde{Z}\left(n\right) = nA + \frac{1}{n}B_E - \frac{1}{n^2}C_E \tag{33}$$

where

$$B_E = \frac{H^2}{2}\frac{d^2}{p}\left[\frac{h}{d}\left(p-d\right)+\pi_1\alpha_0\lambda_1+\pi_2\beta_0\lambda_2+\pi_3\delta_0\lambda_3\right] \tag{34}$$

$$C_E = \frac{d^3H^3}{6p^2}\Big(\pi_1\lambda_1\left[(\lambda_1+2\lambda_2+2\lambda_3)\alpha_0-\alpha_1\alpha_2\right]+\pi_2\lambda_2\left[(2\lambda_1+\lambda_2+2\lambda_3)\beta_0-\beta_1\beta_2\right] + \pi_3\lambda_3\left[\left(\lambda_3{}^2-2\lambda_1\lambda_2\right)\delta_0-\delta_1\delta_2\lambda_3\right]\Big) \quad (35)$$

Again, we easily see that, in this case, the solution approach developed in Section 3 can be used find the optimal solution with B_E and C_E replacing B and C, respectively.

5.3. Numerical Illustration

Example case 1 from Section 4.1 is selected here for the first illustration. Suppose the deteriorating rates are linear with $\alpha_0 = 0.1$, $\beta_0 = 0.1$, $\delta_0 = 0.16$, $\alpha_1 = 0.01$, $\beta_1 = 0.01$, and $\delta_1 = 0.016$, instead of fixed. After calculating B_L and C_L and following Algorithm 1, it is found that $n^* = 2$. When the deteriorating rates are exponential with $\alpha_0 = 0.1$, $\beta_0 = 0.1$, $\delta_0 = 0.16$, $\alpha_1 = 0.01$, $\beta_1 = 0.01$, $\delta_1 = 0.016$, $\alpha_2 = 2$, $\beta_2 = 2$, $\delta_2 = 2$, it is found that $n^* = 3$ as the result of Algorithm 1 using B_E and C_E.

Similar situation is occurred when the same parameters are used for example case 2 from Section 4.2, in which the deteriorating rates are linear with $\alpha_0 = 0.1$, $\beta_0 = 0.1$, $\delta_0 = 0.16$, $\alpha_1 = 0.01$, $\beta_1 = 0.01$, $\delta_1 = 0.016$ and exponential with $\alpha_0 = 0.1$, $\beta_0 = 0.1$, $\delta_0 = 0.16$, $\alpha_1 = 0.01$, $\beta_1 = 0.01$, $\delta_1 = 0.016$, $\alpha_2 = 2$, $\beta_2 = 2$, $\delta_2 = 2$. The n^* value is changed from 4 for fixed deteriorating rates, to 6 for linear deteriorating rates, and 7 for exponential deteriorating rates. While this situation is not taken place for every problem parameters, however, it is implied that linear and exponential deteriorating rates tend to have bigger optimal number of production cycles than the fixed one.

6. Conclusion

In this paper, an economic production quantity (EPQ) model over a finite planning horizon is considered where the production system is imperfect and dictated by two key production subsystems. Each key production subsystem (KPS) will shift from an in-control state to an out-of-control state during a production run, where the time-to-shift of two KPS's are two random variables following a bivariate exponential distribution. When the system is in the out-of-control state, a percentage of items produced become defective and incur additional costs. The objective is to determine a number of production cycles that minimizes the expected total cost per unit time including setup, inventory carrying, and defective costs. Three different defective rate scheme has been investigated, which are fixed, linear, and exponential. It is shown that linear and exponential deteriorating rates tend to have bigger optimal number of production cycles than the fixed one. The future research directions of this model include allowing shortages, considering time value of money, adopting a periodic inspection policy, and investing in setup to improve the quality of the KPS's.

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