ISSN 1816-6075 (Print), 1818-0523 (Online)
Journal of System and Management Sciences
Vol. 12 (2022) No. 2, pp. 303-314
DOI: 10.33168/JSMS.2022.0214

A Study on the Optimal Allocation of Reserve Forces in the (2, 3) Combat Model Using the Lanchester's Square Law

Jong-Hyeon Hwang, Dong-Hyung Lee

Department of Industrial & Management Engineering,
Hanbat National University, Korea
leedh@hanbat.ac.kr

Abstract. This study will propose a plan to optimally allocate additional supportable preliminary combat power in close combat situations (2,3) based on the Lanchester's Square Law. Immediately after the two battle began, they used the Time Zero Allocation of Military Force (TZAMF) scenario, which allocates reserve forces, as soon as they could grasp each other's information and predict the battle pattern. There is a physical distance from the area where the reserve forces are gathered to the battlefield, so it will reflect the travel time, which can increase the possibility of practical use. Various scenarios were set in consideration of the combat situation of the supported units, and differentiated results could be obtained. It is the most traditional, but it can be used as the basic logic of the training using war game model or the decision-making system of the commander.

Keywords: Lanchester's square law, allocation of reserve forces, Time Zero Allocation of Military Force (TZAMF)

1. Introduction

In the actual battle, the question of "how to allocate troops?" is made every moment, and it will be the most important and difficult problem among the decisions made by the commander on the battlefield. Forces refer to various types of combat forces such as people, weapons, and ammunition. The commander who controls multiple battles on multiple battlefields makes decisions while predicting invisible battlefield situations. At this time, it can be said that it is a rather dangerous situation because it relies on the commander's experience and intuition due to a lack of logical evidence using scientific tools.

Because of this importance, there are many studies related to the troop allocation plan. (Roberts et al., 1992) tried to find the minimum force needed to annihilate the defense army, assuming the position of the offensive force commander in the (2,1) battle model in which the two attacking forces and the defense force consisting of only one unit engage. (Kaup 2005) defined this expansion (n, 1) combat model and studied the optimal allocation plan to effectively deal with the attack group from the standpoint of the defense group (Gazijahani et al., 2017). In addition, (Sheeba 2008) proposed an efficient allocation of troops to minimize losses from the standpoint of the basic (2,1) combat model, and tried to reduce the difference in the degree of damage to both battlefields by converting the remaining troops to another battlefield. In addition, there are many studies on the allocation plan of troops (Sheeba 2006; Wu 2009; Krichman 2001). However, there was no decision support study considering the travel time of troops. This study considers the travel time between battlefields of troops to ensure more realistic application conditions.

Lanchester observed air battles between fighters and presented a combat model expressed in a differential equation in which combat power evaluates damage in battles between groups composed of the same weapon system. (Peak & Hong, 2013) predicted the degree of damage and the outcome of the battle of each unit over time with the Lanchester battle model and are still being used steadily to this day. The Lanchester battle model has been studied until recently by various approaches and uses as much as its history and utility. (Kress 2018) applied a new type of model in which three units consume each other, reflecting the modern war pattern in which multilateral conflict could arise as before Syria. (Park 2016) tried to take a more realistic approach by setting combat efficiency as a probability variable. In addition, (Jing 2017) explained that it is effective to concentrate troops in situations where the defense side is not dominant among the combat models of (2,1) using the Battle of Trafalgar, where Britain and France were at odds. This study will use the Lanchester battle model as a rule of engagement between pre-arranged sub-force and the enemy forces that appeared later. It is intended to propose an optimal plan to predict the outcome of the battle and allocate additional troops of the reserve forces to be supported to multiple battlefields. This could practically help commanders make decisions for reserve forces allocation.

There are two types of Lanchester battle models. These are the square models suitable for battles using direct fire weapons and the linear models suitable for battles using grain fire weapons. Often, the two models are mixed to describe guerrilla warfare (Deitchman & Seymour 1962). Also, the probabilistic transformation of combat efficiency, a key element of the model, reflects the uncertainty of combat.

This study defines a (2,3) combat model that includes reserve forces based on the Lanchester Square Law and supports the decision-making of commanders who manage battlefields in an integrated manner. It can be said to be the most basic combat model for allocating troops of the reserve forces, and gradually advanced research has been conducted so that it can be applied to practice through a more complex combat model.

The (2,3) battle model, which includes reserve forces, deals with the battle in two geographically divided battlefields and the situation of reserve forces preparing in other regions to support them. It considers the travel time of troops when applying the TZAMF strategy. Moving reserve forces cannot participate in battles between movements, and the distance traveled to each battlefield is different, so they will have to make decisions by comparing losses and profits.

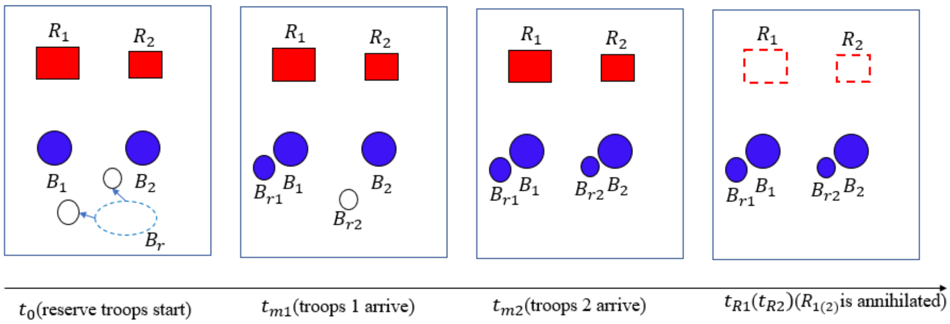

Fig. 1: Time Zero Allocation of Military Force (TZAMF)

This paper consists of four chapters. Chapter 1 explains the background and purpose of the study, and Chapter 2 defines the Lanchester (2,3) battle model and presents it as a mathematical model. Chapter 3 confirms the suitability through objective functions and experiments, and Chapter 4 will conclude.

2. Problem Definition: (2,3) Battle Model

2.1. Lanchester battle model

The Lanchester battle model is a model that describes the two units engaged over time. The equation is expressed in two factors: force size and combat efficiency. In this study, a square model is used among several models. The square model is a model in which combat power is proportional to the square of troops, and the difference in troops acts as a factor that has the greatest influence on the outcome of the battle. Therefore, it was judged that it conforms to the principle of concentration that the concentrated case is more effective than the case where the troops are dispersed, and best describes the modern battle of small units using fire extinguishers. Equation (1) shows the basic unit (1,1) combat model expressed as a square model.

$$\frac{dR_i(t)}{dt} = -\alpha_i B_i(t), \quad R_i(t_0) = M_i \tag{1}$$

$$\frac{dB_i(t)}{dt} = -\beta_i R_i(t), \quad B_i(t_0) = N_i \tag{2}$$

The speed at which the Red team(R) and the Blue team(B) consume each other is determined, and troops consumed is affected by the opposing forces and combat efficiency. The combat efficiency(α) of the Blue Army and the combat efficiency(β) of the Red team are the efficiency of consuming the opponent group. $R_i(t), B_i(t)$ represent the forces of the Red Army and the Blue Army in battlefield i, and the size of the troops of the Blue Army and the Red Army at the start of the Battle of M_i, N_i

2.2. Problem definition

This study assumes the position of the Blue team commander in a situation where the Red team and the Blue team are fighting on two physically divided battlefields (i = 1,2). The Red team is an offensive force, and the Blue Arm which was guarding the battlefield. They do not know the size of each other's troops before the battle but at the start of the battle t_0. To support the battle, the Blue team Reserve (N_r) is located in the rear. When the battle begins and the battle pattern is predicted, it is assigned to the two battlefields ($x_{1,2}$). The moving reserve forces cannot participate in battles between movements, and since the distance to each battlefield is different, the allocation ratio must be determined by predicting the combat patterns of the two battlefields. It is calculated by dividing it by t_0, the time when the reserve forces arrive t_{mi},, and t_{ai} at the end of the battle.

2.3. Purpose of battles

The objective function of the study is set to maximize the survival force for combat purposes. It is an objective function of most existing studies dealing with the Lanchester battle model and one of the most ultimate objectives to be pursued in war.

$$Max \quad Surviving\, Force\, of\, Blue\, team \tag{3}$$

The timing at which the two battlefields end their battles is different, but when they win the battle regardless of the time of end, the scale of survival at the time of end will be calculated.

2.4. Calculating the size of the troops

2.4.1. From the start of the battle, to arrival of the reserve forces ($t_0 \leq t \leq t_{mi}$)

Combats are carried out with existing troops until the reserve forces arrive, and combat patterns can be predicted based on the initial size of troops. Through differential equations of equations (1) and (2) the following equations are derived.

$$B_i(t) = \sqrt{\frac{\beta_i}{\alpha_i}}[-c_i e^{\sqrt{\alpha_i \beta_i}t} + d_i e^{-\sqrt{\alpha_i \beta_i}t}] \tag{4}$$

$$R_i(t) = c_i e^{\sqrt{\alpha_i \beta_i}t} + d_i e^{-\sqrt{\alpha_i \beta_i}t} \tag{5}$$

Where,

$$c_i = -\frac{N_i}{2}\sqrt{\frac{\alpha_i}{\beta_i}} + \frac{M_i}{2}, \qquad d_i = \frac{N_i}{2}\sqrt{\frac{\alpha_i}{\beta_i}} + \frac{M_i}{2} \tag{6}$$

2.4.2. From Arrival of the Reserve Forces, To The End $(t_{mi} \leq t < t_{ai})$

At the time of the arrival of the reserve forces, the size of the troops will increase in addition to the troops in battle. On the other hand, the number of troops in the Red team decreases more rapidly due to the increased number of troops in the Blue team. Equation (7) and (8) show the size of troops in both camps at the time of arrival of the reserve forces in battlefield i.

$$B_i'(t_{mi}) = \sqrt{\frac{\beta_i}{\alpha_i}}\left[-c_i e^{\sqrt{\alpha_i \beta_i}t_{mi}} + d_i e^{-\sqrt{\alpha_i \beta_i}t_{mi}}\right] + x_i \tag{7}$$

$$R_i'(t_{mi}) = c_i e^{\sqrt{\alpha_i \beta_i}t_{mi}} + d_i e^{-\sqrt{\alpha_i \beta_i}t_{mi}} \tag{8}$$

From this point on, the existing equation (4) and (5) cannot be used, and a new equation is required. Equation (9) and (10) represent the size of troops in both camps in battlefield i from the time the reserve forces arrived.

$$B_i'(t) = \sqrt{\frac{\beta_i}{\alpha_i}}[-p_i e^{\sqrt{\alpha_i \beta_i}(t-t_{mi})} + q_i e^{-\sqrt{\alpha_i \beta_i}(t-t_{mi})}] \tag{9}$$

$$R_i'(t) = [p_i e^{\sqrt{\alpha_i \beta_i}(t-t_{mi})} + q_i e^{-\sqrt{\alpha_i \beta_i}(t-t_{mi})}] \tag{10}$$

Where,

$$p_i = -\frac{B_i'(t_{mi})}{2}\sqrt{\frac{\alpha_i}{\beta_i}} + \frac{R_i'(t_{mi})}{2}, \quad q_i = \frac{B_i'(t_{mi})}{2}\sqrt{\frac{\alpha_i}{\beta_i}} + \frac{R_i'(t_{mi})}{2} \tag{11}$$

2.4.3. The end($\mathbf{t} = \boldsymbol{t_{ai}}$)

The end of the battle is t_{ai} when the Red team is annihilated. Using the difference between the arrival of the reserve forces and the annihilation of the Red team, the end of the battle is calculated.

$$t_{ai} - t_{mi} = \frac{1}{2\sqrt{\alpha_i \beta_i}} ln(\frac{q_i}{-p_i}) \quad (12)$$

In addition, in order for this time to exist, the constant in the logarithmic function must be greater than or equal to 1.

$$\frac{q_i}{-p_i} \geq 1 \leftrightarrow -p_i \geq 0 \;\; \rightarrow \;\; \frac{R_i'(t_{mi})}{B_i'(t_{mi})}\sqrt{\frac{\beta_i}{\alpha_i}} \leq 0 \quad (13)$$

When the surviving troops of the Blue team are obtained after the battle is over using Equations (9), (10), and equation (12) is derived,

$$B_i'(t_{ai}) = \sqrt{\frac{\beta_i}{\alpha_i}}\left[2\sqrt{(-p_i)q_i}\right] \quad (14)$$

3. Decision of optimal allocation ratio

3.1. Optimal allocation

$$Max \;\; Z = Surviving \; Force \; of \; Blue \; team \quad (15)$$

If you use equation (14) to carry out equation (16),

$$\begin{aligned} Z &= \sum_i \sqrt{\frac{\beta_i}{\alpha_i}}[2\sqrt{(-p_i)q_i}] \\ &= \sqrt{\frac{\beta_1}{\alpha_1}}\left[2\sqrt{(-p_1)q_1}\right] \\ &+ \sqrt{\frac{\beta_2}{\alpha_2}}[2\sqrt{(-p_2)q_2}] \end{aligned} \quad (16)$$

the DV (Decision Variable) can be said to be an assigned reserve. The reserve units allocated to the first and second battlefields are $x_{1,2}$, and may also be expressed as, $x_2 = N_r - x_1$. If equation (16) is differentiated by x_1 to find the maximum value, then the following equations are derived,

$$\frac{dZ}{dx_1} = (b-d)x^2 + (-2ad - 2bc)x + (bc^2 - a^2 d) \tag{17}$$

Where,

$$\begin{aligned} a &= B_1'(t_{m1}) - x_1, & b &= -\frac{\beta_1}{\alpha_1} R_1'(t_{m1})^2 \\ c &= B_2'(t_{m2}) - (N_r - x_1), & b &= -\frac{\beta_2}{\alpha_2} R_2'(t_{m2})^2 \end{aligned} \tag{18}$$

An x_1 whose derivative value is zero can be obtained using the root formula for solving the quadratic equation.

$$x_1^* = \frac{-(-2ad - 2bc) + \sqrt{(-2ad - 2bc)^2 - 4(b-d)(bc^2 - a^2 d)}}{2(b-d)} \tag{19}$$

3.2. Experiment

To verify the presented mathematical model, the experiment was conducted assuming three situations. It was divided based on the situation of the two battlefields, the first is when the Blue team is advantageous, the second is when one battlefield is disadvantageous, and the last is when both battlefields are disadvantageous. It checks how the decision to allocate reserve forces according to the situation proceeds, and determines whether rational decision-making is possible using the model. For the smooth progress of the experiment, it was assumed that Battlefield 1 was closer to the reserve than Battlefield 2.

Table. 1: Experimental design model

situation	case	Moving unit time
Two blue troops are dominant (range of 1st blue)	105~155(blue) vs 100(red)	10
	120(blue) vs 90(red)	15
Two blue troops are dominant (range of moving time to the 2nd battlefield)	155(blue) vs 100(red)	10
	120(blue) vs 90(red)	15~33
One blue troop is dominant (range of 2nd blue)	130(blue) vs 100(red)	10
	65~85(blue) vs 90(red)	15

One blue troop is dominant (range of moving time to the 2nd battlefield)	130(blue) vs 100(red)	10
	70(blue) vs 90(red)	15~33
No blue troop is dominant (range of 2nd blue)	80(blue) vs 100(red)	10
	65~85(blue) vs 90(red)	15
No blue troop is dominant (range of moving time to the 2nd battlefield)	80(blue) vs 100(red)	10
	70(blue) vs 90(red)	15~33

The experiment observed the optimal solution while changing the parameters of the range. Figure 2 shows the experimental results assuming a situation in which the Blue team prevailed in both battlefields.

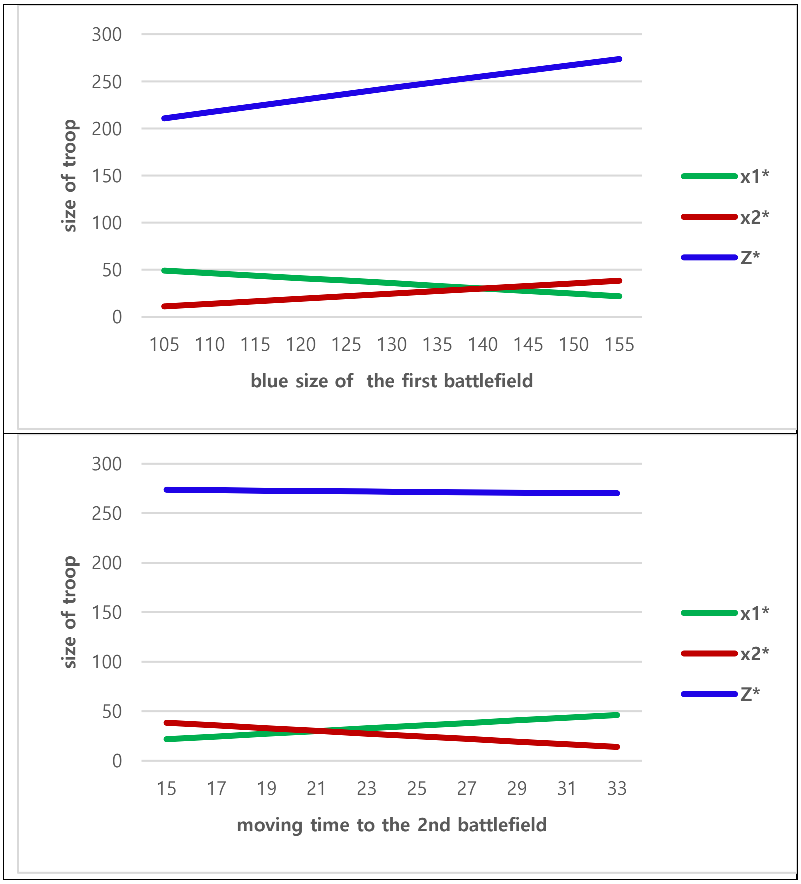

Fig. 2: Two blue troops are dominant.

If both units dominate, reserve forces are assigned to thoroughly maximize the objective function. The upper graph shows the optimal allocation plan by changing the number of troops in the first battlefield, and the lower graph shows the optimal allocation plan by changing the travel time to the second battlefield. It can be seen that nothing special can be found.

Figure 3 shows the experimental results assuming a situation in which the Blue team prevails only in one battlefield.

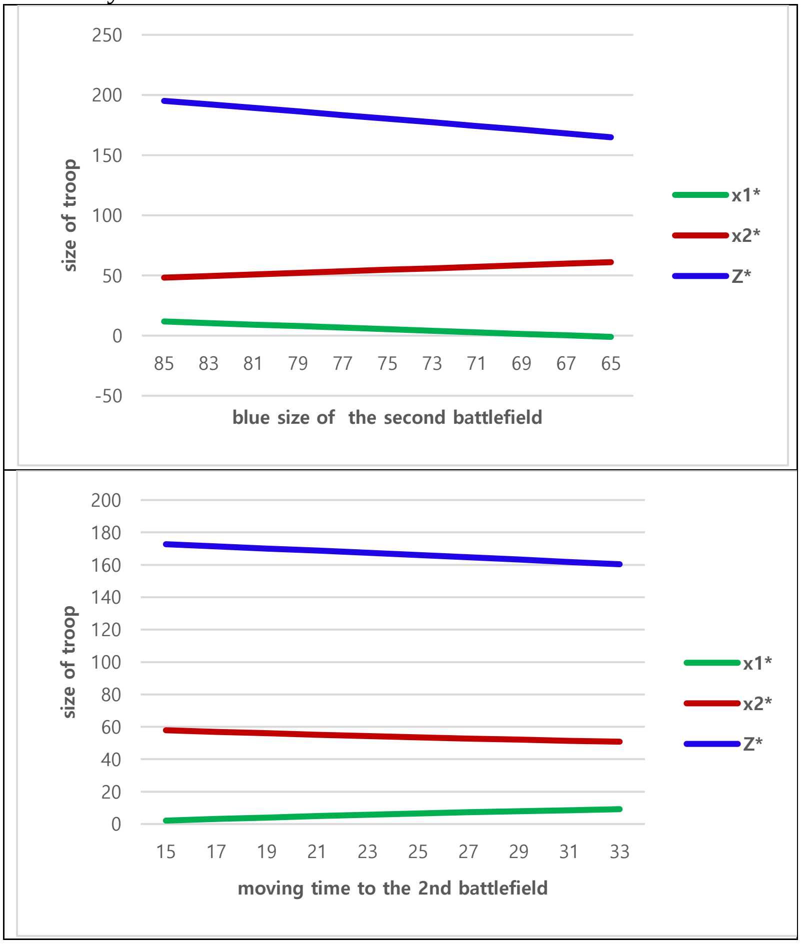

Fig. 3: One blue troop is dominant

This is the result of the experiment on the situation in which the Blue team is advantageous only on the first battlefield. First of all, victory in the two battlefields is important, so the graph at the top shows that the moment B_2becomes 64 units and the reserve forces are not sent to the first battlefield.

Figure 4 shows the experimental results assuming a situation in which the Blue team prevails only in one battlefield.

In situations where both battlefields are at a disadvantage, reserve forces are allocated for victory in both battlefields. The graph at the bottom shows that the time to move to the second battlefield is too long, and there may be cases where the reserve forces are defeated before arrival.

4. Conclusion

In this study, the (2,3) battle model applied with the Lanchester's square law was defined, and optimal allocation plan for the reserve forces was proposed. The travel time of the reserve forces organized to support the troops stationed to defend the base is very important, which is an important factor in decision-making. Through TZAMF, a solution that can be applied to general situations could be obtained, and it was verified through experiments.

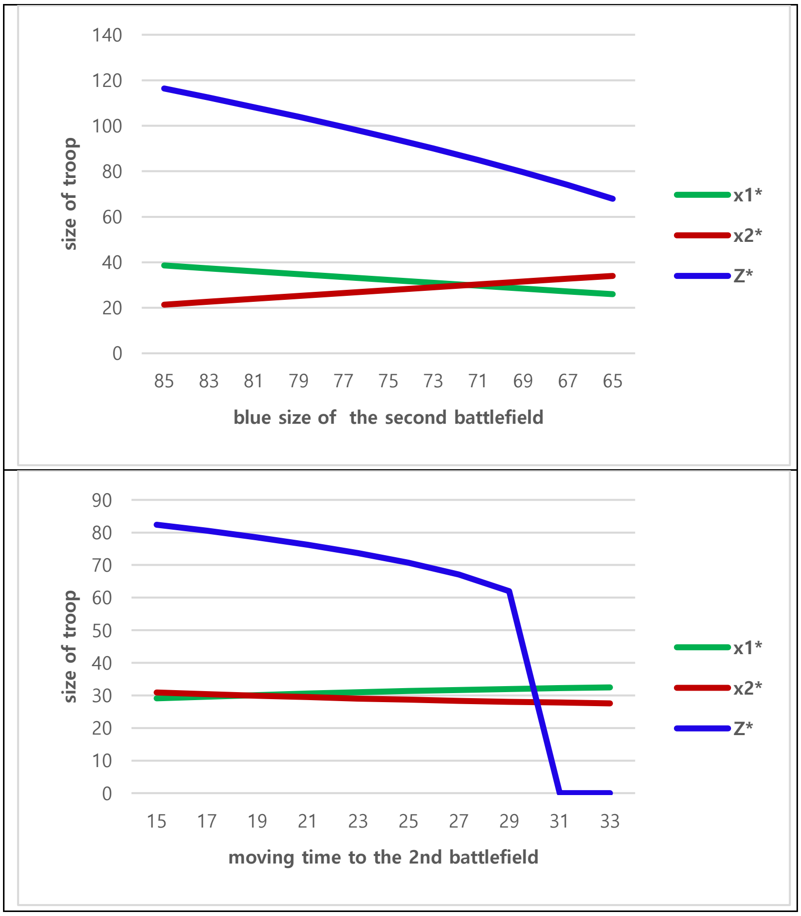

Fig. 4: No blue troops is dominant.

In the future, we will be able to deal with decision-making issues in larger battles so that they can be applied realistically. Using artificial intelligence, it will be possible to find a more complex allocation of detailed weapon systems.

References

Roberts, D. M., & Conolly, B. W. (1992). An extension of the Lanchester square law to inhomogeneous forces with an application to force allocation methodology. *Journal of the Operational Research Society*. 43.8, 741-752.

Kaup, G. T., Kaup, D. J., & Finkelstein, N. M. (2005). The Lanchester (n, 1) problem. *Journal of the Operational Research Society*. 56(12), 1399-1407.

Sheeba, P. S., & Debasish, G. (2008). Optimal resource allocation and redistribution strategy in military conflicts with Lanchester square law attrition. *Naval Research Logistics (NRL)*. 5(6), 581-591.

Sheeba, P. S., & Debasish, G. (2006). Optimal resource partitioning in conflicts based on Lanchester (n, 1) attrition model. *2006 American Control Conference.* IEEE.

Wu, S. –H. & Yang, J. -J. (2009). Optimal military strength allocation for campaign between single-kind arms and multi-kind arms. *2009 International Conference on Management Science and Engineering.* IEEE.

Krichman, M. et al. (2001).Theater level campaign resource allocation. *American Control Conference*. 2001. Proceedings of the 2001. 6. IEEE.

Baek, S. -W. & Hong, S. –P. (2013). A Pragmatic Method on Multi-Weapon Lanchester;s Law. *Journal of the Korean Operations Research and Management Scienve Society*. 38(4), 1-9.

Kress, M. et al. (2018). Lanchester model for three-way combat. *European Journal of Operational Research*. 264(1), 46-54.

Park, D. –H. et al. (2016). Gaussian Approximation of Stochastic Lanchester Model for Heterogeneous Forces. *Journal of Korean Institute of Industrial Engineers*. 42(2), 86-95.

Jing, Y., Zhang, S. D., & Georgi, M. (2017). Complexity of warfare command, communication and control systems simplified: Optimal resource partitioning via Lanchester equations. In: Smart Technologies. *IEEE EUROCON 2017-17th International Conference on*. IEEE, 611-617.

Deitchman & Seymour J. A. (1962). Lanchester model of guerrilla warfare. *Operations Research*. 10(6), 818-827.

Gazijahani, F. S., Abadi, A. A., & Safari, A. (2017). Robust Bi-level Model for Optimal Allocation and Design of Power System Stabilizer in MultiMachine Power Systems. *International Journal of Control and Automation*, NADIA, ISSN: 2005-4297 (Print); 2207-6387 (Online), 10(9), 67-86, http://dx.doi.org/10.14257/ijca.2017.10.9.07.

Kress, M. & Talmor. I. A. (1999). New look at the 3: 1 rule of combat through Markov stochastic Lanchester models. *Journal of the Operational Research Society*. 50(7), 733-744.

Colegrave, R. K., & Hyde, J. M. (1993). The Lanchester square-law model extended to a (2, 2) conflict. *IMA journal of applied mathematics*. 51(2), 95-109.

Park, S. (2021). Coastline Correction Method in Terrain Modeling using Heightmap. *Journal of Next-generation Convergence Technology Association*. 5(6), 965-974.

Lim, K. –T., Lee, H. –I., & Ryu, Y. –S. (2021). A Study on Military Drone Threat Factors and Their Countermeasures. *Journal of Next-generation Convergence Technology Association*. 5(5), 710-720.